Equivalent Statements:

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1. 3.7

The sequence converges. = The sequence has a limit.

The sequence diverges. = The sequence has no limit.

Language: $a_n \to L$ or $\{a_n\} \to L$

"a_n converges to L"

"a_n has limit L"

or $\{a_n\}_{n=1}^{\infty} \to L$

$0! = 1,\ 1! = 1,\ 2! = 2,\ 3! = 6,\ 4! = 24,$

for $n \geq 2$, $n! = n \cdot (n-1)! = n \cdot (n-1) \cdot \dots \cdot 1$.

Important Limits:

For each $\alpha > 0$ (fixed) $$\frac{1}{n^\alpha} \to 0 \quad \text{as} \quad n \to \infty.$$	For each real x (fixed) $$\frac{x^n}{n!} \to 0 \quad \text{as} \quad n \to \infty.$$
If $\lvert x \rvert < 1$, then $$x^n \to 0 \quad \text{as} \quad n \to \infty.$$	If $\varepsilon > 0$ then $$\frac{\ln n}{n^\varepsilon} \to 0 \quad \text{as} \quad n \to \infty.$$ (see 2 pages forward)
If $x > 0$ (fixed), then $$x^{1/n} \to 1 \quad \text{as} \quad n \to \infty.$$	$$n^{1/n} \to 1 \quad \text{as} \quad n \to \infty.$$ $$e^{\ln(n^{1/n})} = e^{\frac{\ln(n)}{n}} \to e^0 = 1$$

For each real x

$$\left(1 + \frac{x}{n}\right)^n \to e^x \quad \text{as} \quad n \to \infty.$$

From chapter 7.

Comment on growth rates

$$\ln(n) << \underset{\varepsilon > 0}{n^{\varepsilon}} << \underset{x > 1}{x^n} << n! << n^n$$

Note: $a_n << b_n$ means

$$\frac{a_n}{b_n} \to 0.$$

$\varepsilon > 0$

$$\frac{\ln n}{n^{\varepsilon}} \to 0 \quad \text{as} \quad \boxed{n \to \infty.} \quad \text{Why?}$$

(from the definition of $\ln(n)$)

First: $\dfrac{\ln(n)}{n} = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{x}\,dx$

Spse $n > 1$.

$$\leq \frac{1}{n}\int_1^n \frac{1}{\sqrt{x}}\,dx$$

b/c $\frac{1}{\sqrt{x}} > \frac{1}{x}$ for $x > 1$.

$$= \frac{1}{n}\, 2\sqrt{x}\,\Big|_1^n = \frac{2}{n}(\sqrt{n} - 1)$$

$$= \frac{2\sqrt{n} - 2}{n} \to 0$$

So, for $n \geq 1$

$$0 \leq \frac{\ln(n)}{n} \leq \frac{2\sqrt{n} - 2}{n} \to 0$$

pinched!

$$\therefore \lim_{n \to \infty} \frac{\ln(n)}{n} = 0.$$

Note: $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{\varepsilon \ln(n)}{\varepsilon n^{\varepsilon}} = \lim_{n \to \infty} \frac{1}{\varepsilon} \frac{\ln(n^{\varepsilon})}{n^{\varepsilon}} \to 0$$

$$= \frac{1}{\varepsilon} \cdot 0 = 0.$$

Example: Give the limit (if it exists) of $\left\{ \frac{\ln(n+1)}{n} \right\}_{n=1}^{\infty} \to 0$

it's "like" $\frac{\ln(n)}{n}$.

$$\frac{\ln(n+1)}{n} = \underbrace{\frac{n+1}{n}}_{\to 1} \cdot \underbrace{\frac{\ln(n+1)}{n+1}}_{\to 0} \to 0.$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n} = 0$$

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2. 4.37

Example: Give the limit (if it exists) of $\left\{\frac{3^n}{4^n}\right\}_{n=1}^{\infty}$. $\longrightarrow 0$

Note: $\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \longrightarrow 0$

$\left|\frac{3}{4}\right| < 1$

$\lim_{n\to\infty} \frac{3^n}{4^n} = 0.$

Example: Give the limit (if it exists) of $\left\{\dfrac{2^{-n}}{3^{-n}}\right\}_{n=1}^{\infty}$ Diverges

$$\frac{2^{-n}}{3^{-n}} = \left(\frac{2}{3}\right)^{-n} = \left(\frac{3}{2}\right)^{n} \to \infty$$

$$\frac{3}{2} > 1$$

$$\lim_{n\to\infty} \frac{2^{-n}}{3^{-n}} = \infty \Rightarrow \text{the limit DNE.}$$

Example: Give the limit (if it exists) of $\left\{ n^{\frac{1}{n+2}} \right\}_{n=1}^{\infty} \to 1$

$$n^{\left(\frac{1}{n+2}\right)} = e^{\ln\left(n^{\frac{1}{n+2}}\right)}$$

$$= e^{\frac{1}{n+2}\ln(n)} = e^{\frac{\ln(n)}{n+2}}$$

$\frac{\ln(n)}{n+2} \to 0$

$$\to e^{0} = 1.$$

Note: Be careful $(2^n)^{1/n} \not\to 1$

$$(2^n)^{1/n} = 2$$

Example: Give the limit (if it exists) of $\left\{\left(1-\frac{1}{n}\right)^{n}\right\}_{n=1}^{\infty} \to \frac{1}{e}$

recall: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$

$\therefore \left(1-\frac{1}{n}\right)^{n} = \left(1+\frac{-1}{n}\right)^{n} \to e^{-1}$

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3. 0

Example: Give the limit (if it exists) of $\left\{\left(1+\frac{3}{n}\right)^n\right\}_{n=1}^{\infty} \to e^3$

State whether the sequence converges and, if it does, find the limit.

2^n. Diverges

$\frac{2}{n}$. $\to 0$

$\frac{(2n+1)^2}{(3n-1)^2}$. $\left(\frac{2n+1}{3n-1}\right)^2$ $\to \frac{4}{9}$

$\ln\left(\frac{2n}{n+1}\right)$. $\to \ln(2)$

$\frac{(-1)^n}{n}$. $\to 0$

$\sqrt{n}$. Diverges

$\frac{n^2}{\sqrt{2n^4+1}}$. $\to \frac{1}{\sqrt{2}}$

$\frac{n^4-1}{n^4+n-6}$. $\to 1$

$\frac{n-1}{n}$. $\to 1$

$\frac{n+(-1)^n}{n}$. $\to 1$

$\cos n\pi$. $= (-1)^n$ diverges

$\frac{n^5}{17n^4+12}$. diverges

$\frac{n+1}{n^2}$. $\to 0$

$\sin\frac{\pi}{2n}$. $\to 0$

$e^{1/\sqrt{n}}$. $= e^{\frac{1}{\sqrt{n}}}$ $\to 1$

$\sqrt{4-\frac{1}{n}}$. $\to 2$

$\frac{2^n}{4^n+1}$. $\to 0$

$\frac{n^2}{n+1}$. diverges

$\ln n - \ln(n+1)$. $\to 0$

$\frac{2^n-1}{2^n}$. $\to 1$

$(-1)^n\sqrt{n}$. diverges

$\frac{4n}{\sqrt{n^2+1}}$. $\to 4$

$\frac{\sqrt{n+1}}{2\sqrt{n}}$. $\to \frac{1}{2}$

$\frac{1}{n}-\frac{1}{n+1}$. $\to 0$

$(-\frac{1}{2})^n$ $\to 0$

$\frac{4^n}{2^n+10^6}$. diverges

$\tan\frac{n\pi}{4n+1}$. $\to \tan(\pi/4) = 1$

$\frac{10^{10}\sqrt{n}}{n+1}$. $\to 0$

$\left(1+\frac{1}{n}\right)^{2n}$ $\to e^2$

$\left(1+\frac{1}{n}\right)^{n/2}$ $\to e^{1/2}$

$\frac{2^n}{n^2}$. diverges

$2\ln 3n - \ln(n^2+1)$. $\to \ln(9)$