Equivalent Statements:

The sequence converges. = The sequence has a limit.

The sequence diverges. = The sequence has no limit.

$n! \equiv$ "n factorial" $\quad 0! = 1,\quad 1! = 1,$

$2! = 2\cdot 1,\quad 3! = 3\cdot 2\cdot 1,$

$4! = 4\cdot 3\cdot 2\cdot 1, \ldots, n! = n(n-1)!$

Important Limits:

For each $\alpha > 0$ ← fixed $$\frac{1}{n^{\alpha}} \to 0 \quad \text{as} \quad n \to \infty.$$	For each real x ← fixed $$\frac{x^n}{n!} \to 0 \quad \text{as} \quad n \to \infty.$$
If $\lvert x\rvert < 1$, then ← fixed $$x^n \to 0 \quad \text{as} \quad n \to \infty.$$	If $\varepsilon > 0$, then $$\frac{\ln n}{n^{\varepsilon}} \to 0 \quad \text{as} \quad n \to \infty.$$ (see next page.)
If $x > 0$, then ← fixed $$x^{1/n} \to 1 \quad \text{as} \quad n \to \infty.$$	$$n^{1/n} \to 1 \quad \text{as} \quad n \to \infty.$$ $e^{\ln(n^{1/n})} = e^{\frac{\ln(n)}{n}} \to e^0 = 1$

$\boxed{(2^n)^{1/n}}$

For each real x ← Fixed

$$\left(1 + \frac{x}{n}\right)^n \to e^x \quad \text{as} \quad n \to \infty.$$

From chapter 7.

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$\frac{\ln n}{n} \to 0 \quad \text{as} \quad \boxed{n \to \infty.} \quad \text{Why?}$$

(from the definition of ln(n))

$$\frac{1}{n}\ln(n) = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$$

Assume $n \geq 1$

$1 \leq x \leq n$

$\Rightarrow \sqrt{x} \leq x$

$\Rightarrow \frac{1}{\sqrt{x}} \geq \frac{1}{x}$.

$$\leq \frac{1}{n}\int_1^n \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{1}{n}\, 2\sqrt{x}\,\Big|_1^n = \frac{1}{n}\left(2\sqrt{n} - 2\right)$$

$$\therefore \quad 0 \leq \frac{\ln(n)}{n} \leq \frac{2\sqrt{n} - 2}{n} \to 0 \quad \text{as } n \to \infty$$

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$$\therefore \quad \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

Now let $\varepsilon > 0$.

$$\lim_{n\to\infty} \frac{\varepsilon \ln(n)}{\varepsilon\, n^{\varepsilon}} = \lim_{n\to\infty} \frac{1}{\varepsilon} \cdot \boxed{\frac{\ln(n^{\varepsilon})}{n^{\varepsilon}}} = 0.$$

$\to 0$

$$\therefore \quad \frac{\ln(n)}{n^{\varepsilon}} \to 0.$$

Question:

We know

$$\lim_{n\to\infty} \frac{\ln(n)}{n^{.0000001}} = 0.$$

How large must n become before we see $n^{.000001}$ beginning to overtake $\ln(n)$?

$$\frac{\ln(n)}{n^{.0000001}} = 10^{7} \frac{\ln\left(n^{.0000001}\right)}{n^{.0000001}} \quad \leftarrow x$$

$$= 10^{7} \frac{\ln(x)}{x} \qquad \text{let } x = 10^{14}$$

$$= 10^{7} \frac{\ln(10^{14})}{10^{14}}$$

$$= 10^{-7} \cdot \underbrace{14 \cdot \ln(10)}_{\approx 32.24} \quad \leftarrow \text{small}$$

Need $n^{.0000001} \geq 10^{14} \Rightarrow n \geq \left(10^{14}\right)^{10^{7}}$

Comments on Growth Rates

$$\ln(n) \ll n^{\varepsilon} \ll x^{n} \ll n! \ll n^{n}$$

$\varepsilon > 0$ Fixed $x > 1$ Fixed

<u>Note</u>: when I write $a_n \ll b_n$,

I mean $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$.

Example: Give the limit (if it exists) of $\left\{ \dfrac{\ln(n+1)}{n} \right\}_{n=1}^{\infty} \to 0$

$$\frac{\ln(n+1)}{n} = \boxed{\frac{n+1}{n}} \cdot \boxed{\frac{\ln(n+1)}{n+1}} \to 0$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n} = 0 .$$

Example: Give the limit (if it exists) of $\left\{\frac{3^n}{4^n}\right\}_{n=1}^{\infty} \to 0$

$$\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \longrightarrow 0$$

$$0 < \frac{3}{4} < 1$$

$$\lim_{n\to\infty} \frac{3^n}{4^n} = 0.$$

Example: Give the limit (if it exists) of $\left\{\frac{2^{-n}}{3^{-n}}\right\}_{n=1}^{\infty}$ Diverges

$$\frac{2^{-n}}{3^{-n}} = \left(\frac{2}{3}\right)^{-n} = \left(\frac{3}{2}\right)^{n} \longrightarrow \infty$$

$$\frac{3}{2} > 1$$

Diverges b/c $\lim_{n\to\infty} \frac{2^{-n}}{3^{-n}} = \infty$.

Example: Give the limit (if it exists) of $\left\{ n^{\frac{1}{n+2}} \right\}_{n=1}^{\infty}$. $\to 1$

$$n^{\frac{1}{n+2}} = e^{\ln\left(n^{\frac{1}{n+2}}\right)} = e^{\frac{\ln(n)}{n+2}} \to 1$$

$$\frac{\ln(n)}{n+2} \to 0$$

$$\lim_{n\to\infty} n^{\frac{1}{n+2}} = 1.$$

<u>Be careful:</u> $\left(2^n\right)^{1/n} \not\to 1$

$$\left(2^n\right)^{1/n} = 2$$

Example: Give the limit (if it exists) of $\left\{\left(1-\frac{1}{n}\right)^n\right\}_{n=1}^{\infty} \to \frac{1}{e}$

Recall: $\lim_{x\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$\therefore \left(1-\frac{1}{n}\right)^n = \left(1+\frac{-1}{n}\right)^n \to e^{-1}$

Example: Give the limit (if it exists) of $\left\{\left(1+\frac{3}{n}\right)^{n}\right\}_{n=1}^{\infty} \rightarrow e^{3}$

State whether the sequence converges and, if it does, find the limit.

2^n. diverges

$\dfrac{2}{n}$. $\to 0$

$\left(\dfrac{2n+1}{3n-1}\right)^2 \to \left(\dfrac{2}{3}\right)^2$

$\dfrac{(2n+1)^2}{(3n-1)^2}$. $\to \dfrac{4}{9}$

$\ln\left(\dfrac{2n}{n+1}\right)$. $\to \ln(2)$; $\dfrac{2n}{n+1} \to 2$

$\dfrac{(-1)^n}{n}$. $\to 0$

$\sqrt{n}$. diverges

$\dfrac{n^2}{\sqrt{2n^4+1}}$. $\to \dfrac{1}{\sqrt{2}}$

$\dfrac{n^4-1}{n^4+n-6}$. $\to 1$

$\dfrac{n-1}{n}$.

$\dfrac{n+(-1)^n}{n}$.

$\cos n\pi$. $=(-1)^n$ diverges

$\dfrac{n^5}{17n^4+12}$. diverges

$\dfrac{n+1}{n^2}$. $\to 0$

$\sin\left(\dfrac{\pi}{2n}\right) \to 0$; $\dfrac{\pi}{2n} \to 0$

$\dfrac{2^n}{4^n+1}$. $\to 0$

$\dfrac{n^2}{n+1}$. diverges

$e^{1/\sqrt{n}}$. $\to e^0 = 1$

$\sqrt{4-\dfrac{1}{n}}$. $\to \sqrt{4} = 2$

$(-1)^n\sqrt{n}$. diverges

$\dfrac{4n}{\sqrt{n^2+1}}$. $\to 4$

$\ln n - \ln(n+1)$. $\to 0$; $\ln\left(\dfrac{n}{n+1}\right)$

$\dfrac{2^n-1}{2^n}$. $\to 1$

$(-\frac{1}{2})^n \to 0$; $|-\frac{1}{2}|<1$

$\dfrac{4^n}{2^n+10^6}$. diverges

$\dfrac{\sqrt{n+1}}{2\sqrt{n}}$. $\to \dfrac{1}{2}$

$\dfrac{1}{n} - \dfrac{1}{n+1}$. $\to 0$

$\tan\left(\dfrac{n\pi}{4n+1}\right)$. $\to 1$; $\dfrac{n\pi}{4n+1} \to \pi/4$

$\dfrac{10^{10}\sqrt{n}}{n+1}$. $\to 0$

$\left(1+\dfrac{1}{n}\right)^{2n} \to e^2$

$\left(1+\dfrac{1}{n}\right)^{n/2} \to e^{1/2}$

$\dfrac{2^n}{n^2}$. diverges

$2\ln 3n - \ln(n^2+1)$.

$\ln((3n)^2) - \ln(n^2+1)$

$= \ln\left(\dfrac{9n^2}{n^2+1}\right)$; $\dfrac{9n^2}{n^2+1} \to 9$

$\to \ln(9)$