Info...

Test 3 review videos are posted!

We will have a review for Test 3 on Friday from 4-6pm.

Examine the limits below.

\[
limit_{x \to \infty} \frac{\sin(x) - x}{x^2} = \frac{0}{0}
\]

\[
limit_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = 4
\]

These are examples of 0/0 indeterminant forms.

Why is 0/0 called an indeterminant form?

\[
limit_{x \to 1} \frac{17(x - 1)}{x - 1} = 17
\]

\[
limit_{x \to 0} \frac{\sin(2x)}{5x} = \frac{2}{5}
\]

\[
limit_{x \to \infty} \frac{17(x - 1)}{x^2 - 1} = \frac{17}{\infty} = 0
\]

\[
limit_{x \to 0} \frac{\sin(x)}{x} = 1
\]

Bottom Line: "0/0" can lead to anything.

What is an (\(\infty/\infty\)) indeterminant form?

Example:

\[
limit_{x \to \infty} \frac{3x - 2}{2x + 1} = \frac{3}{2}
\]

\[
limit_{x \to \infty} \frac{\tan\left(\frac{\pi}{2}x\right)}{\ln(x - 1)}
\]
Why is $\frac{\infty}{\infty}$ called an indeterminant form?

**Answer:** It can be almost anything!

\[
\begin{align*}
\lim_{x \to \infty} \frac{3x-2}{2x+1} &= \infty \\
\lim_{x \to \infty} \frac{x^2}{x+1} &= \infty \\
\lim_{x \to \infty} \frac{x^2}{x^3+1} &= 0
\end{align*}
\]

**DNE.**

L'Hopital's Rule...

\[
\lim_{x \to 0} \frac{f(x)}{g(x)}
\]

Two Cases:
1. $(0/0)$
2. $(\infty/\infty)$

These are both "indeterminant forms."

**L'Hopital's Rule (00) Case.**

**Theorem:** Suppose that $f$ and $g$ are differentiable functions with $g(x) \neq 0$, and suppose

\[
\begin{align*}
f(x) &\to 0 \text{ and } g(x) \to 0 \\
&\text{as } x \to c, c' \in \mathbb{R}, \text{ or } -\infty, \infty.
\end{align*}
\]

If $\frac{f'(x)}{g'(x)} \to L$ then $\frac{f(x)}{g(x)} \to L$.

i.e.

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

...when we have a 0/0 indeterminant form!!

**Note:** $\lim_{x \to 1} \frac{3x-2}{2x+1}$ This is not a $0/0$ indeterminant form.

\[
\begin{align*}
\lim_{x \to 1} \frac{3x-2}{2x+1} &= \frac{3(1)-2}{2(1)+1} \\
&= \frac{1}{3}.
\end{align*}
\]

**But, if you are too smart to try substitution, you might try L'Hopital's rule (legally). This leads to the wrong answer of $\frac{3}{2}$.**

Why does this work? **See the video.**

Since $f(x) \to 0$ and $g(x) \to 0$ as $x \to c$, we can assume $f'(c) = g'(c) = 0$.

Pick any value $b < c$, and define $h(x) = f(x) - \frac{f(b)}{g(b)} g(x)$.

**Note:** $h(b) = h'(c) = 0$.
L’Hospital’s Rule: \((\infty/\infty)\) Case.

**Theorem** Suppose that \(f\) and \(g\) are differentiable functions with \(g'(x) \neq 0\), and suppose

\[ f(x) \to \pm \infty \text{ and } g(x) \to \pm \infty \]

as \(x \to c\), \(c\), \(c\), or \(-\infty\). Let \(L\) be a real number or \(\pm \infty\).

If \(\frac{f'(x)}{g'(x)} \to L\) then \(\frac{f(x)}{g(x)} \to L\).

i.e.

\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \]

...provided we have an \(\infty/\infty\) indeterminant form!!!

**Example:**

\[ \lim_{x \to 0} \frac{2\cos(x) - 2 + 3x^2}{x^3} = \frac{2}{3} \]

\(\text{ind. form} \)

\[ \text{Try L-H: } \lim_{x \to 0} \frac{-2\sin(x) + 6x}{2x} = 3 \text{ ind. form} \]

\[ \text{Try L-H again!} \]

\[ \lim_{x \to 0} \frac{-2\cos(x) + 6}{2} = 2 \]

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**Example:**

\[ \lim_{x \to \infty} \frac{\sin(x)}{e^x} = 0 \]

\(\text{we know that } e^x \text{ grows faster than any polynomial of } x\) as \(x \to \infty\).

\(\text{But, suppose you are brain dead.}\)

\[ \frac{\sin(x)}{e^x} \text{ ind. form. } \text{Try L-H: } \lim_{x \to \infty} \frac{\sin(x)}{e^x} = 0 \]

\(\text{try ind. form} \)

\[ \text{Try L-H again! } \lim_{x \to \infty} \frac{2}{e^x} = 0 \]
Example: \[
\lim_{x \to \infty} \frac{\ln(x)}{x} = 0
\]
we know this. In fact, \( \ln(x) \) grows slower than \( \text{any positive power of } x \).

But, if you don't recall this, note that we have an \( \frac{\infty}{\infty} \) indeterminate. 

Try L'Hôpital: \[
\lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} 1 = 0
\]

Example: \[
\lim_{x \to \infty} \frac{\ln(x)}{x^k} = 0 \quad \text{b/c } \ln(x) \text{ grows slower (eventually) than any positive power of } x.
\]

But, if you don't recall this, note that it is in \( \frac{\infty}{\infty} \) indeterminate form. 

Try L'Hôpital: 

\[
\lim_{x \to \infty} \frac{1/x}{kx^{k-1}} = \lim_{x \to \infty} \frac{1}{kx^{k-1}} = 0
\]

What is wrong with the following argument?

\[
\lim_{x \to 0} \frac{x^2}{\sin(x)} = \lim_{x \to 0} \frac{2x}{\cos(x)} = \lim_{x \to 0} \frac{2}{\sin(x)} = -\infty
\]

Not an ind. form.

Popper 19

1. 1  2. 2  3. 3  4. 4  5. 5
Examples:

\[
\lim_{x \to 0} \left( e^x + 2x \right)^{1/x} = e^3
\]

Different types of ind. forms.

\[
\lim_{x \to \infty} \frac{\ln(e^x + 2x)}{x} = e^3
\]

??

Look at

\[
\lim_{x \to 0^+} \frac{\ln(e^x + 2x)}{x}
\]

\(0/0\)

Ind. form

Try L'Hôpital:

\[
\lim_{x \to 0^+} \frac{e^x + 2}{e^x + 2x} = 3
\]