Examine the limits below.

"$\frac{0}{0}$" indeterminant form

$$\underline{\lim_{x\to 0}} \frac{\sin(x)-x}{x^3}$$

Not so easy.

"$\frac{0}{0}$" indeterminant form

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \underline{\underline{4}}$$

These are examples of
0/0 indeterminant forms.

Why is 0/0 called an indeterminant form?

"$\frac{0}{0}$" $$\lim_{x\to 1}\frac{17(x-1)}{x-1} = 17$$

$$\lim_{x\to 0}\frac{\sin(2x)}{5x} = \frac{2}{5}$$ "$\frac{0}{0}$"

"$\frac{0}{0}$" $$\lim_{x\to 1}\frac{17(x-1)}{\underbrace{x^2-1}_{(x-1)(x+1)}} = \frac{17}{2}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$ "$\frac{0}{0}$"

Bottom Line: "0/0" can lead to anything.

What is an (∞/∞) indeterminant form?

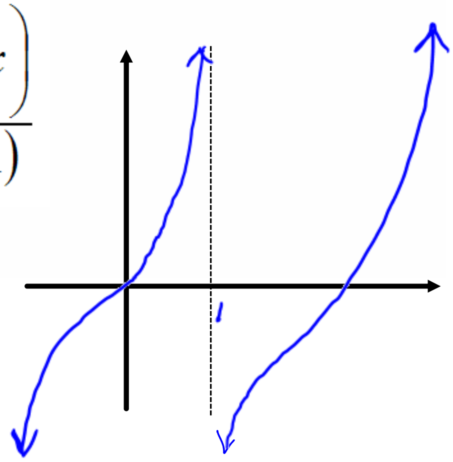
Example:

"$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty}\frac{3x-2}{2x+1} = \frac{3}{2}$$

"$\frac{-\infty}{-\infty}$"

$$\lim_{x\to 1^+}\frac{\tan\left(\frac{\pi}{2}x\right)}{\ln(x-1)}$$



"$\frac{\infty}{\infty}$", "$\frac{-\infty}{\infty}$", "$\frac{-\infty}{-\infty}$", "$\frac{\infty}{-\infty}$"

Why is (∞/∞) called an indeterminant form?

Answer: It can be almost anything!

$$\lim_{x\to\infty}\frac{3x-2}{2x+1} = \frac{3}{2}$$

$$\lim_{x\to\infty}\frac{x^2}{x+1} = \infty$$

DNE

$$\lim_{x\to\infty}\frac{x^2}{x^3+1} = 0$$

L'Hospital's Rule...

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Two Cases:

1. $(0/0)$
2. (∞/∞)

These are both "indeterminant forms."

L'Hospital's Rule: (0/0) Case.

Theorem *Suppose that f and g are differentiable functions with $g'(x) \neq 0,$ and suppose*

$$f(x) \to 0 \text{ and } g(x) \to 0$$

as $x \to c^+, c^-, c, \infty, -\infty.$ Let L be a real number or $\pm\infty$.

$$\text{If } \frac{f'(x)}{g'(x)} \to L \text{ then } \frac{f(x)}{g(x)} \to L$$

"$\frac{0}{0}$" ind. form

i.e.

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

...when we have a 0/0 indeterminant form!!

Cation: "$\frac{0}{0}$" ind form is important.

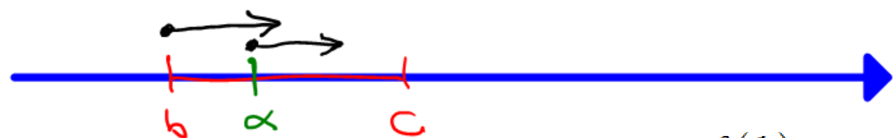
ex. $\lim_{x\to 0} \frac{3x-1}{2x+6} = -\frac{1}{6}.$

Note: If you wrongly apply L-H rule (b/c we do not have a "$\frac{0}{0}$" ind. form), then you get $\frac{3}{2}$. WRONG.

Why does this work?

Since $f(x) \to 0$ and $g(x) \to 0$ as $x \to c$, we can assume

$f(c) = g(c) = 0$. Assume f, g are diff.



Pick any value $b < c$, and define $h(x) = f(x) - \underbrace{\frac{f(b)}{g(b)} g(x)}$.

Note: $h(b) = h(c) = 0$

$$h(b) = f(b) - \frac{f(b)}{g(b)} g(b) = 0$$

$$h(c) = \overset{=0}{f(c)} - \frac{f(b)}{g(b)} \overset{=0}{g(c)} = 0$$

∴ by Rolle's Thm there is α so that $b < \alpha < c$ and $h'(\alpha) = 0$.

Note: $h'(x) = f'(x) - \frac{f(b)}{g(b)} g'(x)$

Then $h'(\alpha) = 0$ iff $f'(\alpha) - \frac{f(b)}{g(b)} g'(\alpha) = 0$

i.e. $\frac{f'(\alpha)}{g'(\alpha)} = \frac{f(b)}{g(b)}$.

So, when $b \to c^-$, we get $\alpha \to c^-$, and

$\frac{f'(\alpha)}{g'(\alpha)} \to L \Rightarrow \frac{f(b)}{g(b)} \to L$.

L'Hospital's Rule: (∞/∞) Case.

Theorem *Suppose that f and g are differentiable functions with $g'(x) \neq 0,$ and suppose*

$$f(x) \to \pm\infty \text{ and } g(x) \to \pm\infty$$

"$\frac{\infty}{\infty}$" ind. form

as $x \to c^+, c^-, c, \infty, -\infty.$ Let L be a real number or $\pm\infty$.

$$\text{If } \frac{f'(x)}{g'(x)} \to L \text{ then } \frac{f(x)}{g(x)} \to L$$

i.e.

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

...provided we have an ∞/∞ indeterminant form!!!

Example: $\lim_{x\to 0} \dfrac{2\cos(x)-2+3x^2}{x^2} = 2$

"$\frac{0}{0}$" ind. form.

~~Try~~ L-H : $\lim_{x\to 0} \dfrac{-2\sin(x)+6x}{2x} = 2$ New limit problem

"$\frac{0}{0}$" ind form

Try L-H $\lim_{x\to 0} \dfrac{-2\cos(x)+6}{2} = 2$

$\lim_{x\to 0}\left(-\dfrac{\sin(x)}{x} + 3\right) = 2$

Revisit...

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3} = -\frac{1}{6}$$

"$\frac{0}{0}$" ind. form

Try L-H: $\lim_{x\to 0}\frac{\cos(x)-1}{3x^2} = -\frac{1}{6}$

New limit problem

"$\frac{0}{0}$" ind. form AGAIN

Try L-H: $\lim_{x\to 0}\frac{-\sin(x)}{6x} = \lim_{x\to 0} -\frac{1}{6}\cdot\frac{\sin(x)}{x}$

$\to 1$

$= -\frac{1}{6}$

Example: $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$ $= 0$ ← we know this.

But, let's suppose you aren't thinking.

"$\frac{\infty}{\infty}$" ind. form.

Try L-H: $\lim\limits_{x\to\infty} \dfrac{2x}{e^x} = 0$

"$\frac{\infty}{\infty}$" ind. form

Try L-H: $\lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$

Example: $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ ← you know this.

BUT, let's suppose you aren't thinking.

"$\frac{\infty}{\infty}$" ind. form.

Try L-H: $\lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

Example: $\lim_{x\to\infty} \frac{\ln(x)}{x^{0.0000001}}$ $= 0$ you know this.

<u>But</u>, if you aren't thinking ...

"$\frac{\infty}{\infty}$" ind. form

Try L-H: $\lim_{x\to\infty} \frac{1/x}{0.0000001\, x^{(0.0000001-1)}}$

$= \lim_{x\to\infty} \frac{1}{0.0000001\, x^{0.0000001}} = 0$

What is wrong with the following argument?

$$\lim_{x\to 0^+} \frac{x^2}{\sin(x)} = \overset{0}{\overset{\parallel}{\lim_{x\to 0^+} \frac{2x}{\cos(x)}}} = \lim_{x\to 0^+} \frac{2}{-\sin(x)} = \cancel{-\infty}$$

Check ind. forms?

$$\text{“}\frac{0}{0}\text{”} \checkmark \qquad \text{“}\frac{0}{1}\text{”}$$

not an ind. form.

$\Rightarrow$ this is invalid.

Example: $\lim_{x\to 0^+} \left(\underbrace{e^x + 2x}_{\to 1}\right)^{1/x \to \infty} = e^3$ "1^∞" ind. form.

Note
$$(e^x+2x)^{1/x} = e^{\ln\left((e^x+2x)^{1/x}\right)}$$
$$= e^{\boxed{\frac{\ln(e^x+2x)}{x}}} \to e^{\boxed{3}}$$

$\lim_{x\to 0^+} \frac{\ln(e^x+2x)}{x}$

"$\frac{0}{0}$" ind. form

Try L-H: $\lim_{x\to 0^+} \frac{\frac{e^x+2}{e^x+2x}}{1} = \frac{3}{1} = 3$