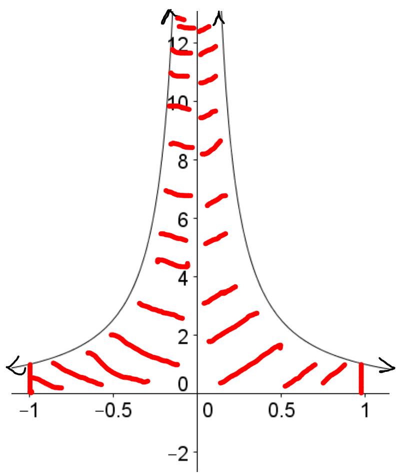
Improper Integrals

should be > 0

Motivating Example: $\int_{-1}^{1} x^{-4/3}\,dx \cancel{=} -3x^{-1/3}\Big|_{-1}^{1}$

$= -3\left[1 - (-1)^{-1/3}\right]$

$\boxed{= -6}$ oops!



Problem

Trouble: $x^{-4/3}$ is not continuous on $[-1, 1]$. So, the F Thm Calc does not apply.

Remark: **The previous integral is referred to as an "Improper" Integral.**

Types of Improper Integrals

$$\int_a^b f(x)dx \qquad \int_{-\infty}^b f(x)dx \qquad \int_a^\infty f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx$$

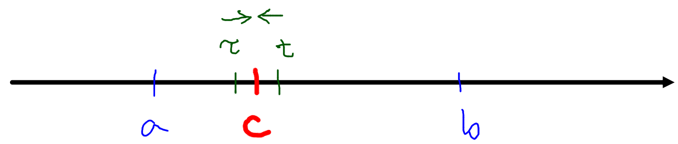
where a limit of integration is infinite and/or the function f has a discontinuity on the interval.

Proper Notation/Computation

$$\int_a^b f(x)dx$$

Assume f is continuous except at $x = c$, where c lies in the interval $[a,b]$.

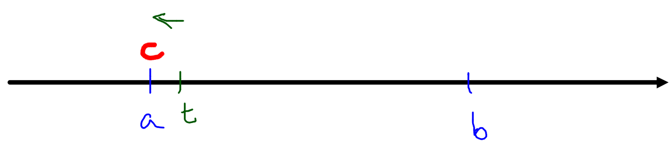
Case 1:



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

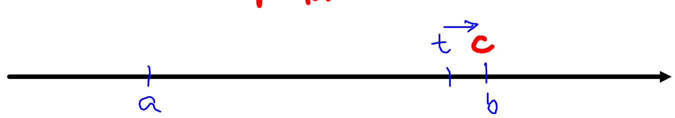
$$= \lim_{\tau \to c^-} \underbrace{\int_a^{\tau} f(x)\,dx}_{\text{F Thm Calc ok}} + \lim_{t \to c^+} \underbrace{\int_t^b f(x)\,dx}_{\text{ditto}}$$

Case 2:



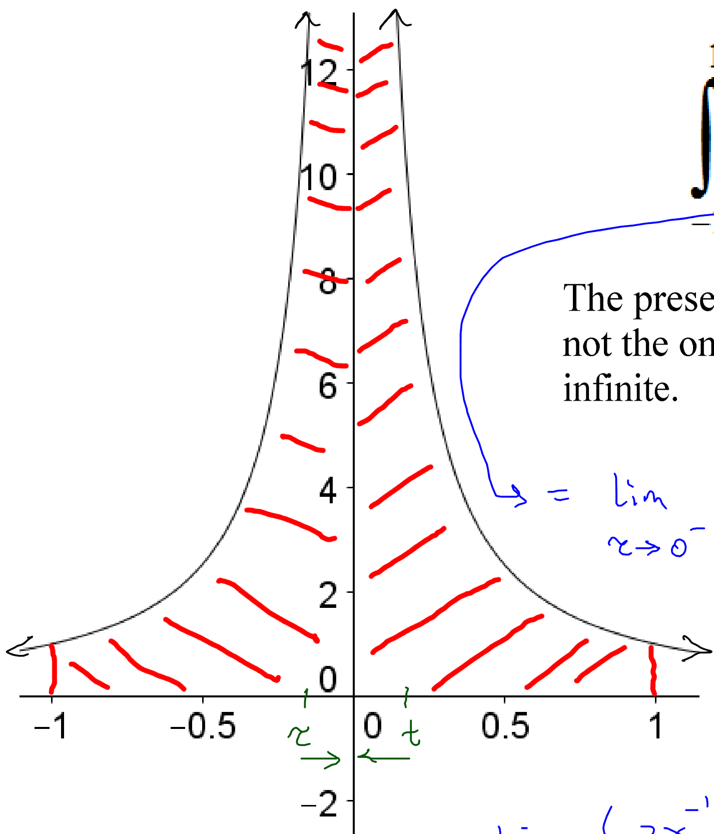
$$\int_a^b f(x)\,dx = \lim_{t \to c^+} \underbrace{\int_t^b f(x)\,dx}_{\text{F Thm Calc ok}}$$

Case 3:



Similar

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \underbrace{\int_a^t f(x)\,dx}_{\text{F Thm Calc ok}}$$



$$\int_{-1}^{1} x^{-4/3}\,dx = \underline{\underline{DNE}}$$

The presence of the vertical asymptote is not the only reason why this integral is infinite.

$$= \lim_{\tau\to 0^-} \underbrace{\int_{-1}^{\tau} x^{-4/3}\,dx}_{\text{F. Thm Calc ok}} + \lim_{t\to 0^+} \underbrace{\int_{t}^{1} x^{-4/3}\,dx}_{\text{F. Thm Calc ok}}$$

$$= \lim_{\tau\to 0^-} \left(-3x^{-1/3}\right)\Big|_{-1}^{\tau} + \lim_{t\to 0^+} \left(-3x^{-1/3}\right)\Big|_{t}^{1}$$

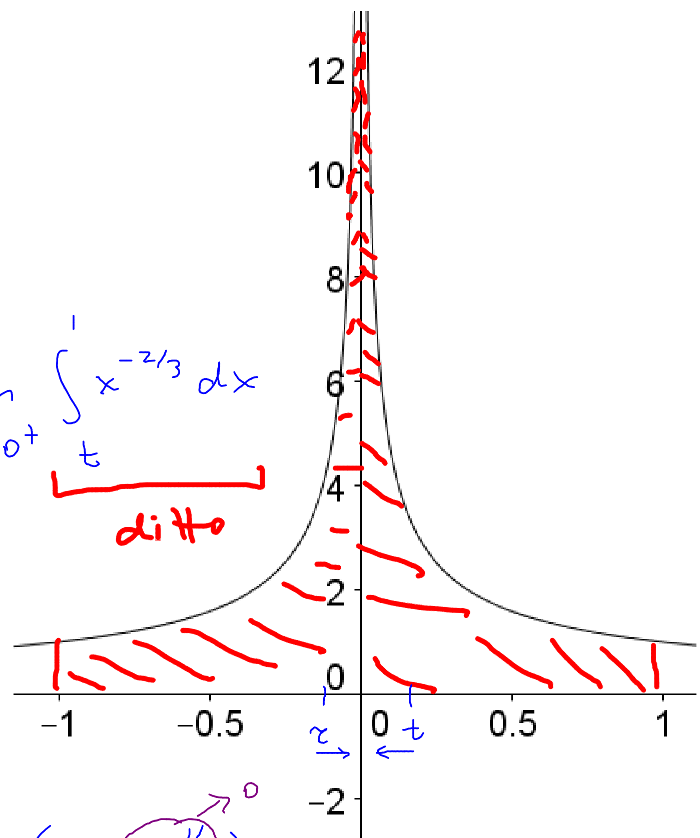
$$= \lim_{\tau\to 0^-} \left(-3\underbrace{\frac{1}{\tau^{1/3}}}_{\to -\infty} - 3\right) + \lim_{t\to 0} \left(-3 + 3\underbrace{\frac{1}{t^{1/3}}}_{\to \infty}\right)$$

$$= \infty$$

Similar Example

$$\int_{-1}^{1} x^{-2/3}\,dx = 6$$

$$= \lim_{\tau\to 0^-}\underbrace{\int_{-1}^{\tau} x^{-2/3}\,dx}_{\text{F. Thm Calc ok}} + \lim_{t\to 0^+}\underbrace{\int_{t}^{1} x^{-2/3}\,dx}_{\text{ditto}}$$



$$= \lim_{\tau\to 0^-} 3x^{1/3}\Big|_{-1}^{\tau} + \lim_{t\to 0^+} 3x^{1/3}\Big|_{t}^{1}$$

$$= \lim_{\tau\to 0^-}\left(\overset{\to 0}{3\tau^{1/3}} + 3\right) + \lim_{t\to 0^+}\left(3 - \overset{\to 0}{3t^{1/3}}\right) = 6$$

How do we handle

$$\int_{\boxed{-\infty}}^{\boxed{\infty}} f(x)\,dx, \quad \int_{\boxed{-\infty}}^{b} f(x)\,dx, \quad \int_{a}^{\boxed{\infty}} f(x)\,dx$$

when f is continuous?

$$\int_{a}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

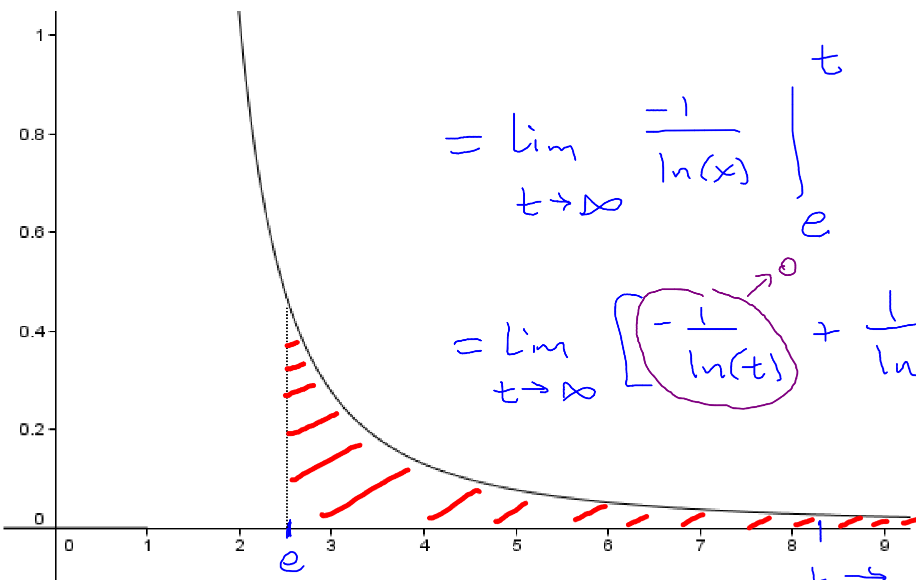
$$\lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\lim_{t\to-\infty} \int_t^0 f(x)\,dx + \lim_{t\to\infty} \int_0^t f(x)\,dx$$

Example: $$\int_e^{\boxed{\infty}} \frac{1}{x\left(\ln(x)\right)^2}\,dx = \lim_{t\to\infty} \underbrace{\int_e^t \frac{1}{x\left(\ln(x)\right)^2}\,dx}_{\text{F. Thm Calc ok}}$$

$$= \lim_{t\to\infty} \left.\frac{-1}{\ln(x)}\right|_e^t$$

$$= \lim_{t\to\infty}\left[\overbrace{-\frac{1}{\ln(t)}}^{\to 0} + \frac{1}{\ln(e)}\right] = 1$$



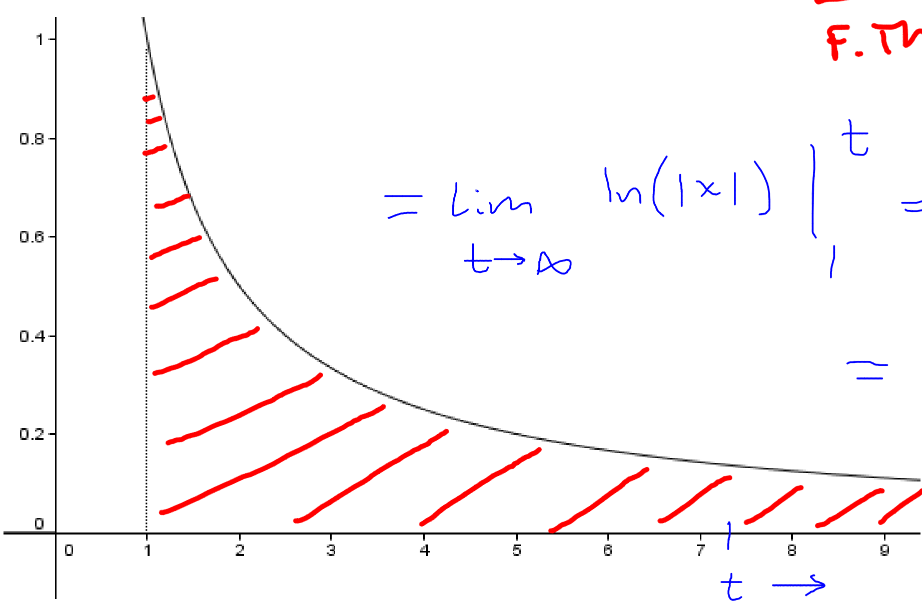
A portion of the graph of $\dfrac{1}{x(\ln(x))^2}$.

Example: $\displaystyle\int_1^{\boxed{\infty}} \frac{1}{x}\,dx = \text{DNE}$

$$= \lim_{t\to\infty} \underbrace{\int_1^t \frac{1}{x}\,dx}_{\text{F. Thm Calc}} \quad \text{ok}$$

$$= \lim_{t\to\infty} \ln(|x|)\Big|_1^t = \lim_{t\to\infty} \left(\ln(t) - 0\right)$$

$$= \infty$$

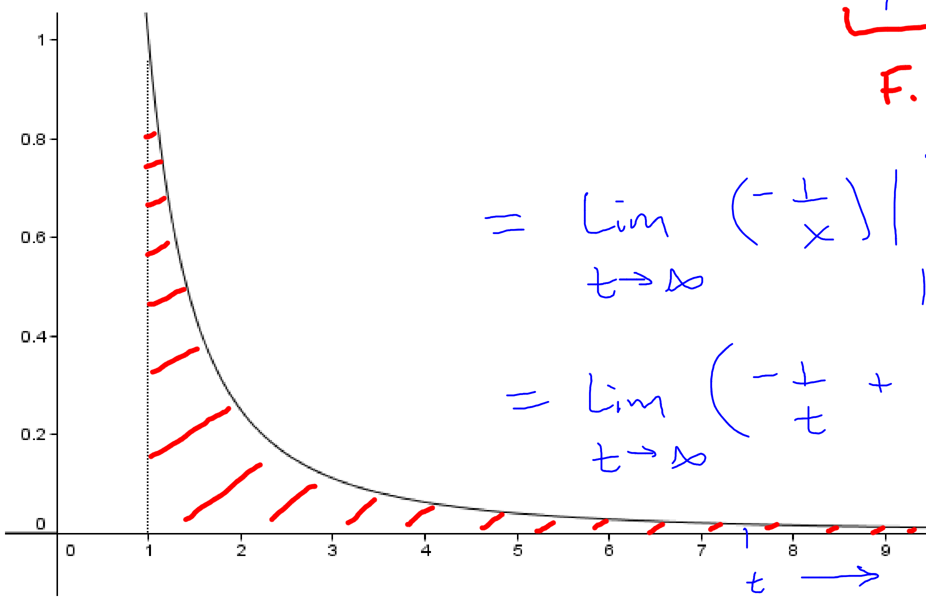


A portion of the plot of $y = 1/x$.

Example: $\displaystyle\int_1^{\boxed{\infty}} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \underbrace{\int_1^t \frac{1}{x^2}\,dx}_{\text{F. Thm Calc OK}}$

$$= \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1$$



A portion of the plot of $y = 1/x^2$.

An Important Improper Integral...

$$\int_1^{\boxed{\infty}} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^{-p}}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \begin{cases} \lim\limits_{t\to\infty} \ln(x)\Big|_1^t & p = 1 \\ \lim\limits_{t\to\infty} \dfrac{1}{-p+1} x^{-p+1}\Big|_1^t & p \neq 1 \end{cases}$$

$$= \begin{cases} \infty & p = 1 \\ \lim\limits_{t\to\infty}\left[\dfrac{1}{1-p}t^{1-p} - \dfrac{1}{1-p}\right] & p \neq 1 \end{cases}$$

$1-p>0$

$1>p$

$$= \begin{cases} \infty & p = 1 \\ \infty & p < 1 \\ -\dfrac{1}{1-p} = \dfrac{1}{p-1} & p > 1 \end{cases}$$

$$\therefore \int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{DNE} & p \leq 1 \\ \dfrac{1}{p-1} & p > 1 \end{cases}$$

(b/c ∞)