Recall: Infinite Series

Sums

terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

could be anything

Relation to the sequence of partial sums:

S_N = sum of the first N terms

$\{S_N\}_{N=1}^{\infty} \equiv$ seq. of partial sums

N^{th} partial sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

Note: For $\sum_{n=3}^{\infty} b_n$, we have

$$S_1 = b_3 \quad , \quad S_2 = b_3 + b_4 \; , \; \ldots$$

$$\sum_{n=3}^{\infty} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=3}^{N+2} b_n$$

Popper 23:

$$\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$$

1. $S_1 =$

2. $S_2 =$

3. $S_3 =$

4. $S_{10} =$

Explorations

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Last time

Observations based on computed partial sums:

N	S_N
999988	14.392715
999989	14.392716
999990	14.392717
999991	14.392718
999992	14.392719
999993	14.39272
999994	14.392721
999995	14.392722
999996	14.392723
999997	14.392724
999998	14.392725
999999	14.392726
1000000	14.392727

Possible Conclusions:

You might "think" that the seq. of partial sums is converging.

i.e. you might conclude that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a finite value.

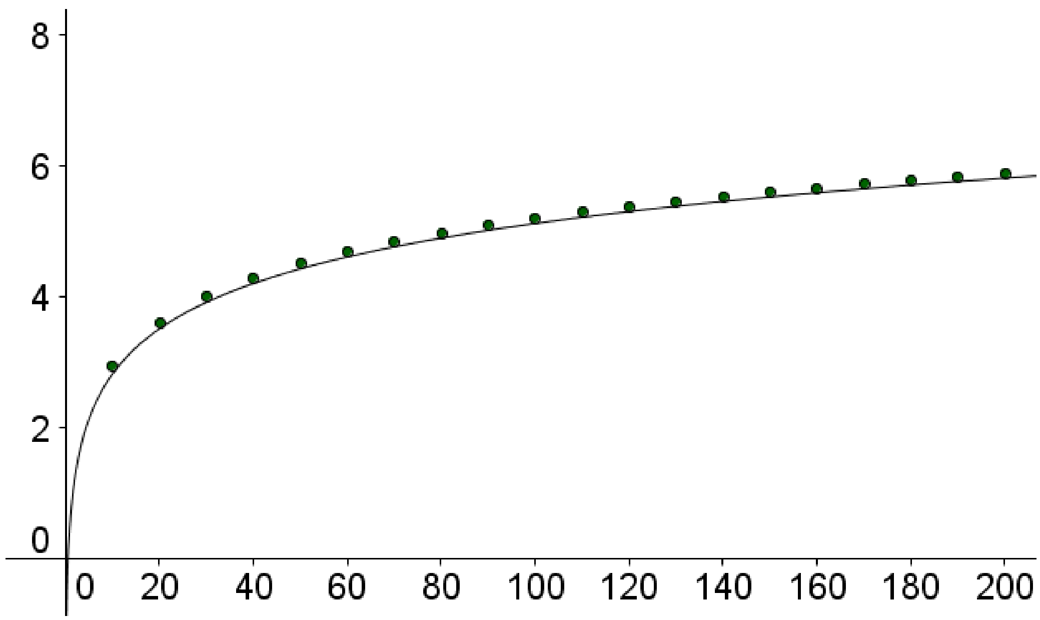
But this is wrong.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

i.e. DNE

Plot of S_N vs $f(N) = \ln(N) + 0.5$ for

$$\sum_{n=1}^{\infty} \frac{1}{n}$$



An Observation

The infinite series below have positive terms.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges}$$

Questions: What does this tell us about the sequence of partial sums?

It is increasing.

What can we say about an increasing sequence that is bounded above?

It converges.

What can we say about an increasing sequence that is not bounded above?

Diverges.

$\therefore$ If $a_n > 0$ for all n, then

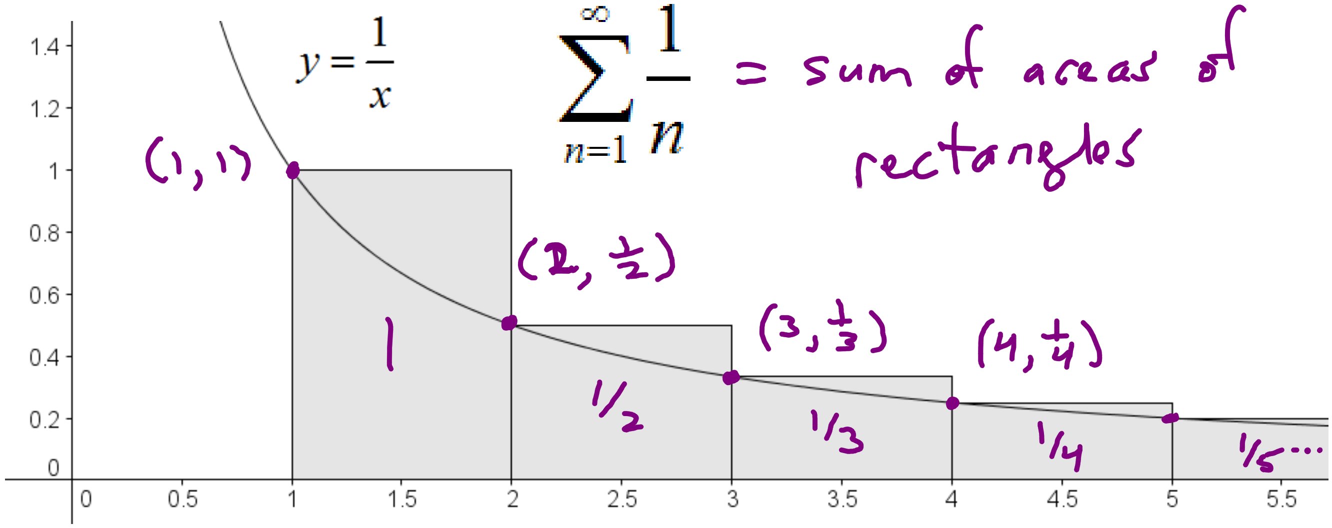
$\sum_{n=?}^{\infty} a_n$ converges iff

$\sum_{n=?}^{\infty} a_n$ is bounded.

i.e. $\{S_N\}_{N=1}^{\infty}$ is bounded.

Interpret the Following

$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{sum of areas of rectangles}$$



Conclusion?

Note: The area btwn $\frac{1}{x}$ and the x-axis for $1 \le x < \infty$ is less than $\sum_{n=1}^{\infty} \frac{1}{n}$.

i.e. $$\int_1^{\infty} \frac{1}{x}\,dx \le \sum_{n=1}^{\infty} \frac{1}{n}$$

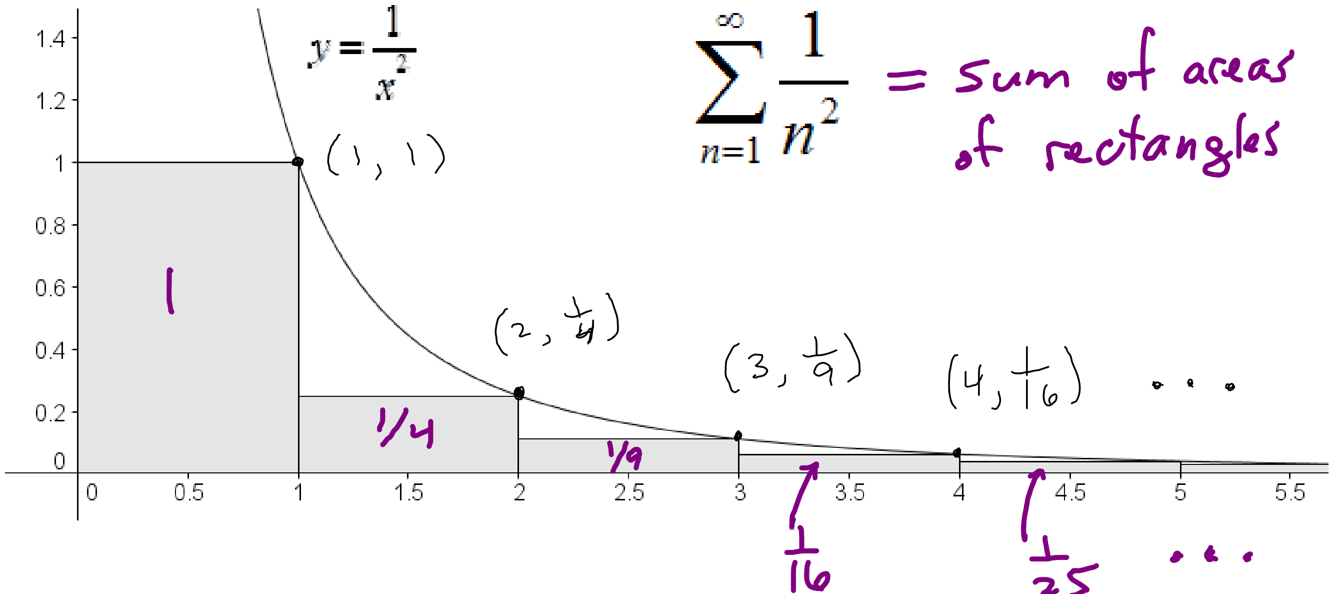
$$\| \qquad \qquad \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

∞

DNE

b/c it is infinite

Interpret the Following



$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ = sum of areas of rectangles

Conclusion?

Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$

$\leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$

$= 1 + \frac{1}{2-1} = 2$

i.e. $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$ and the terms are positive, so the series converges.

Note:
$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}}_{p\text{-series}} = \underbrace{\begin{cases} \text{diverges if } p \le 1 \\ \text{converges if } p > 1 \end{cases}}_{p\text{-series test.}}$$

ex.
$$\sum_{n=3}^{\infty} \frac{1}{n^4}$$
converges b/c it is a p-series with $p = 4 > 1$.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$
diverges b/c it is a p-series with $p = 1/2 \le 1$.

A General Observation for Infinite Series

$$\sum_{n=1}^{\infty} a_n = \text{sum of infinitely many terms.}$$

Question: What if the terms of a series do not go to zero?

The series Diverges.

Answer: **(divergence test) The series diverges.**

Divergence Test Flow Chart:

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} a_n = 0 \text{ ??}$$

do the terms go to zero?

No

Series Diverges!!

Yes

The series might converge, or it might diverge.

Get to Work!!

Example: $$\sum_{n=1}^{\infty} \frac{n}{1000n+1}$$

Terms:

$$\frac{n}{1000n+1} \to \frac{1}{1000} \neq 0$$

$\therefore$ The series diverges.

Example: $$\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n}$$

Terms: $\left(1-\frac{1}{n}\right)^{n} \to \frac{1}{e} \neq 0$

$\therefore$ the series diverges.

We can typically only determine whether an infinite series (sum) converges or diverges. Quite often, we cannot find the actual sum.

Some exceptions:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$$

PFD

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

Easy

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$$

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

$$1 = A(n+3) + B(n+2)$$

Kinen n's : $n = -3$

$$1 = -B \qquad \boxed{B = -1}$$

$n = -2$

$$\boxed{1 = A}$$

$\therefore$

$$\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} + \frac{-1}{n+3}$$

So,
$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{(n+2)(n+3)}$$

$$= \lim_{N \to \infty} \left[\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{N+2} - \frac{1}{N+3}\right) \right]$$

$$= \lim_{N \to \infty} \left[\frac{1}{3} - \frac{1}{N+3} \right] = \frac{1}{3}$$

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{3}{2}.$$

Seq of partial sums

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{3}$$

$$S_3 = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2$$

$$\vdots$$

$$S_N = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N$$

$$\times \quad \left(1 - \frac{1}{3}\right)$$

$$\left(1 - \frac{1}{3}\right) S_N = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N - \frac{1}{3}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N\right)$$

$$= 1 - \left(\frac{1}{3}\right)^{N+1}$$

$$S_N = \frac{1 - \left(\frac{1}{3}\right)^{N+1}}{1 - \frac{1}{3}} \to \frac{1}{1 - \frac{1}{3}}$$

Geometric Series: Suppose r is a given real number.

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \text{Diverges} & \text{otherwise} \end{cases}$$

Example: $$\sum_{n=4}^{\infty} \frac{(-1)^n}{3^n}$$

Next Time...