Recall: Infinite Series

Sums

terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

could be anything

Relation to the sequence of partial sums:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

$\vdots$

$\{S_N\}_{N=1}^{\infty}$

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

Explorations

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Last time

This might converge.

We will see that it actually does!

Observations based on computed partial sums:

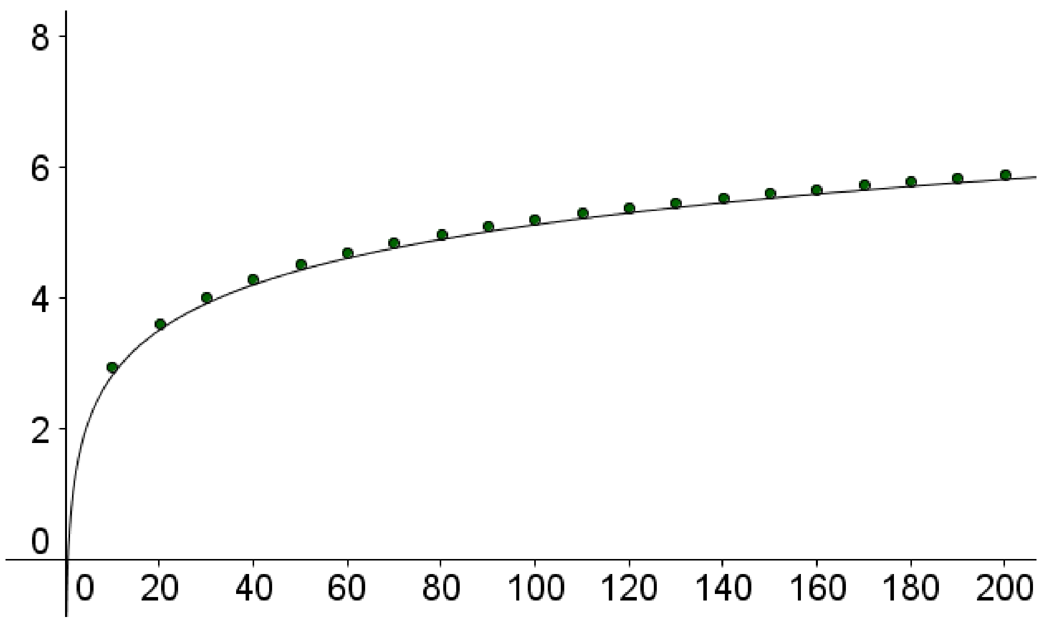
n	$1/n$	S_n
99981	1.00019E-05	12.08995611
99982	1.00018E-05	12.08996611
99983	1.00017E-05	12.08997612
99984	1.00016E-05	12.08998612
99985	1.00015E-05	12.08999612
99986	1.00014E-05	12.09000612
99987	1.00013E-05	12.09001612
99988	1.00012E-05	12.09002612
99989	1.00011E-05	12.09003612
99990	1.0001E-05	12.09004613
99991	1.00009E-05	12.09005613
99992	1.00008E-05	12.09006613
99993	1.00007E-05	12.09007613
99994	1.00006E-05	12.09008613
99995	1.00005E-05	12.09009613
99996	1.00004E-05	12.09010613
99997	1.00003E-05	12.09011613
99998	1.00002E-05	12.09012613
99999	1.00001E-05	12.09013613
100000	0.00001	12.09014613

Possible Conclusions:

From looking at these values, it is not unreasonable to think that this series converges. **Unfortunately, this is wrong!**

Plot of S_N vs $f(N) = \ln(N) + 0.5$ for

$$\sum_{n=1}^{\infty} \frac{1}{n}$$



An Observation

The infinite series below have positive terms.

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Questions: What does this tell us about the sequence of partial sums?

Increasing.

What can we say about an increasing sequence that is bounded above?

It converges.

What can we say about an increasing sequence that is not bounded above?

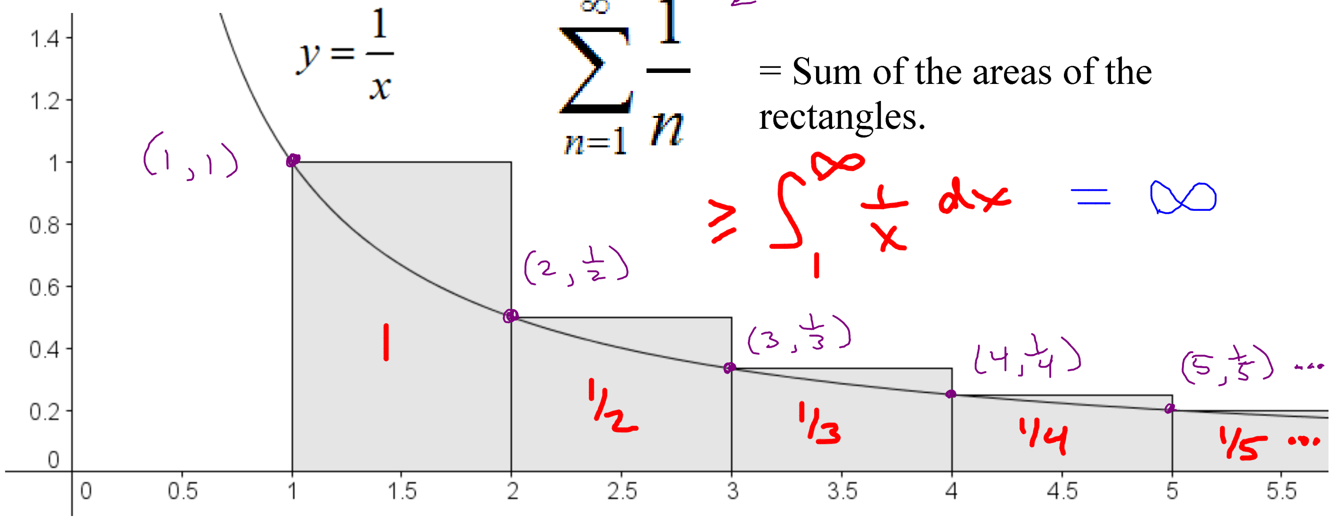
Diverges

Interpret the Following

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ = Sum of the areas of the rectangles.

$$\geq \int_1^{\infty} \frac{1}{x}\,dx = \infty$$

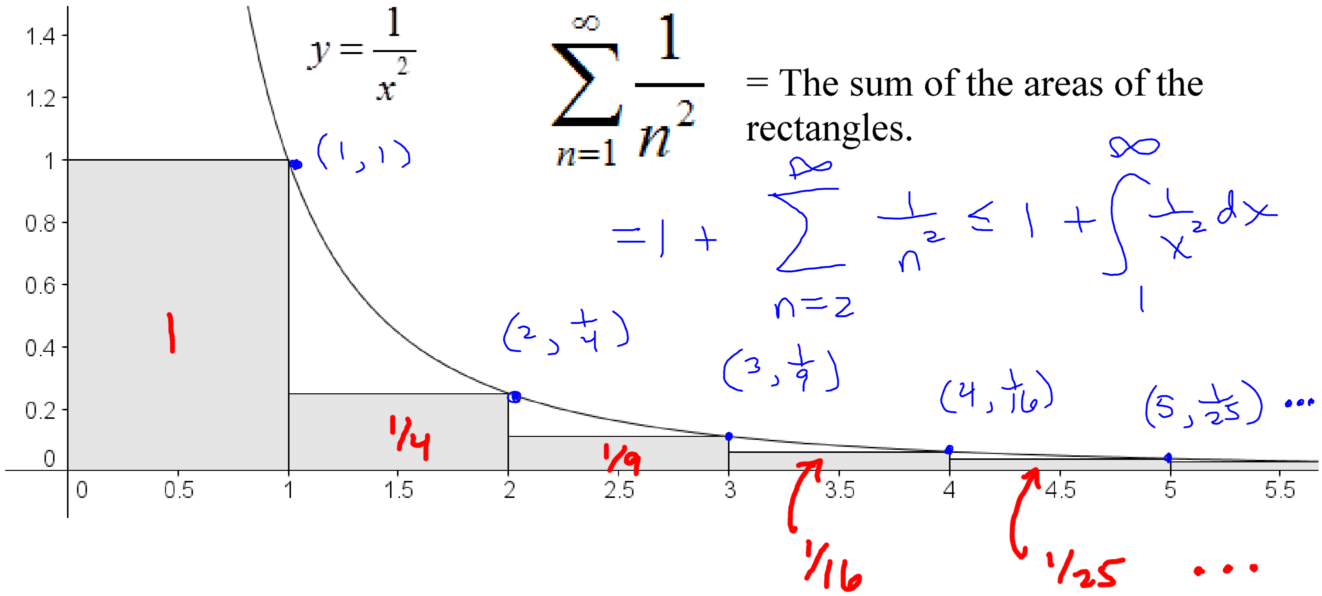


Conclusion?

$\sum_{n=1}^{\infty} \frac{1}{n}$ is infinite

$\therefore$ it diverges

Interpret the Following



$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ = The sum of the areas of the rectangles.

$$= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \le 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

Conclusion?

we know $\int_1^{\infty} \frac{1}{x^2}\,dx$ is finite.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite.

positive

Consequently, our sequence of partial sums is increasing and bounded above. Therefore, it converges.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

It is possible to show this is $\pi^2/6$.

Similarly,

p-series

p-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges}, & p > 1 \\ \text{diverges}, & p \le 1 \end{cases}$$

whatever

Examples: $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p series with $p = \frac{3}{2} > 1$

$\therefore$ it converges

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p series with $p = \frac{1}{2} \le 1$

$\therefore$ it diverges

A General Observation for Infinite Series

$$\sum_{n=1}^{\infty} a_n \quad = \text{ Sum of infinitely many terms}$$

Question: What if the terms of a series do not go to zero?

$$\sum_{n=1}^{\infty} a_n \text{ must diverge.}$$

Answer: (divergence test) The series diverges.

Divergence Test Flow Chart:

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} a_n = 0 \text{ ??}$$

Do the terms go to zero?

No

Series Diverges!!

Yes

The series might converge, or it might diverge.

Get to Work!!

Recall: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, although the terms go to 0.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1000n+1}$

Terms $\dfrac{n}{1000n+1} \to \dfrac{1}{1000}$

$\neq 0$

Since the terms do not go to zero, the series diverges.

Example: $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n}$

Terms: $\left(1-\frac{1}{n}\right)^{n} \to \frac{1}{e}$

$\neq 0$

Since the terms do not go to zero, the series diverges.

We can typically only determine whether an infinite series (sum) converges or diverges. Quite often, we cannot find the actual sum.

Some exceptions:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3}$$

use PFD

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{3}{2}$$

Easy

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3}$$

PFD

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

$$1 = A(n+3) + B(n+2)$$

Killer n's: $n=-3$, $n=-2$

$n=-3$: $1 = -B$

$n=-2$: $1 = A$

PFD

$$\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} + \frac{-1}{n+3}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \lim_{N\to\infty} \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{N+2} - \frac{1}{N+3}\right]$$

$$= \lim_{N\to\infty} \left[\frac{1}{3} - \frac{1}{N+3}\right] = \frac{1}{3}.$$

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{3}{2}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \lim_{N \to \infty} \sum_{n=0}^{N} \left(\frac{1}{3}\right)^n$$

$$= \lim_{N \to \infty} \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N \right]$$

$$= \lim_{N \to \infty} \frac{\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N\right)\left(1 - \frac{1}{3}\right)}{1 - \frac{1}{3}}$$

$$= \lim_{N \to \infty} \frac{\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N\right) - \left(\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^N + \left(\frac{1}{3}\right)^{N+1}\right)}{1 - \frac{1}{3}}$$

$$= \lim_{N \to \infty} \frac{1 - \left(\frac{1}{3}\right)^{N+1}}{1 - \frac{1}{3}} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

Geometric Series: Suppose r is a given real number.

similarly

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

Geom. Series Test

$$\underbrace{\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r} = \begin{cases} \frac{1}{1-r}, & |r|<1 \\ \text{diverges} & |r|\geq 1 \end{cases}}$$

Example: $\sum_{n=4}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=4}^{\infty} \left(-\frac{1}{3}\right)^n$

$= \sum_{n=4}^{\infty} \left(-\frac{1}{3}\right)^4 \left(-\frac{1}{3}\right)^{n-4}$

$= \left(-\frac{1}{3}\right)^4 \sum_{n=4}^{\infty} \left(-\frac{1}{3}\right)^{n-4}$

$= \left(-\frac{1}{3}\right)^4 \sum_{m=0}^{\infty} \left(-\frac{1}{3}\right)^m$

Note: $\left|-\frac{1}{3}\right| < 1$

$= \frac{1}{81} \cdot \frac{1}{1 - -\frac{1}{3}}$

$= \frac{1}{81} \cdot \frac{3}{4} = \frac{1}{108}$

$= \frac{1}{108}$