Review:

Infinite Series...

Sums

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Terms

Could start anywhere.

Relation to the sequence of partial sums:

$\{S_N\}_{N=1}^{\infty}$

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

S_N = sum of the first N terms

Review:

Terms

$$\sum_{n=1}^{\infty} a_n$$

Question: What if the terms of a series do not go to zero?

Answer: (divergence test) The series diverges.

Review:

Flow Chart

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} a_n = 0 \text{ ??}$$

Do the terms go to zero.

No

Series Diverges!!

Yes

Get to Work!!

Recall:

We can typically only determine whether an infinite series (sum) converges or diverges. Quite often, we cannot find the actual sum.

Some exceptions:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$$

Last time, we used PFD to show this sum is 1/3.

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1} \left(\frac{1}{3}\right)^n$$

$$= \lim_{N\to\infty} \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{N-1}\right) \frac{\left(1-\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \lim_{N\to\infty} \frac{\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{N-1}\right) - \left(\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{N-1} + \left(\frac{1}{3}\right)^N\right)}{1-\frac{1}{3}}$$

$$= \lim_{N\to\infty} \frac{1-\left(\frac{1}{3}\right)^N}{1-\frac{1}{3}} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$= \frac{3}{2}$$

New:

Geometric Series: Suppose r is a given real number.

$$\boxed{\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad \text{if } r \neq 1.}$$

$$\overbrace{\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}}^{\text{Geom. Series Test}}$$

Why? Mock the previous page.

More Generally...

fixed

$$\sum_{n=m}^{\infty} r^n = \sum_{n=m}^{\infty} r^m \cdot r^{n-m}$$

Fixed

0, 1, 2, 3, ...

$$= r^m \sum_{n=m}^{\infty} r^{n-m}$$

$$= r^m \boxed{\sum_{k=0}^{\infty} r^k} = \begin{cases} \dfrac{r^m}{1-r} & |r|<1 \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

Review: Infinite Series With Nonnegative Terms

$$\sum_{n=1}^{\infty} a_n, \quad \text{with } a_n \geq 0 \text{ for all } n.$$

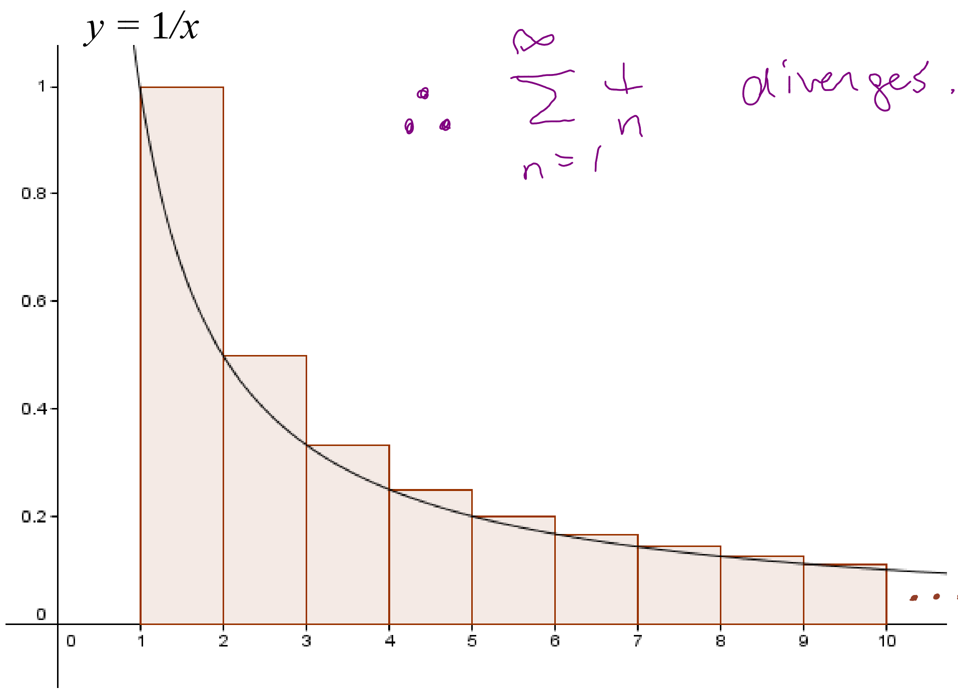
Note: In this case, what can we always say the sequence of partial sums is nondecreasing (typically, increasing).

And... a sequence that is increasing and bounded above has a limit.

Therefore, if $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ for all n, and the sum is bounded above, then it converges. Also, if the sum is not bounded above, then it diverges.

Recall: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx = \infty$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.



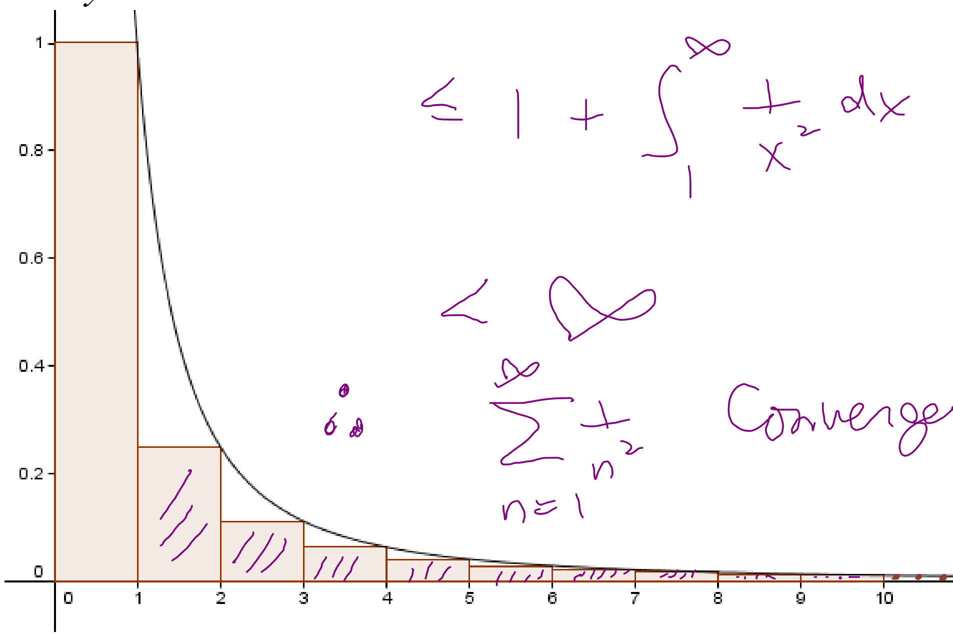
Recall: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ $\displaystyle = 1 + \sum_{n=2}^{\infty}\frac{1}{n^2}$

$$\le 1 + \int_1^{\infty}\frac{1}{x^2}\,dx$$

$$< \infty$$

$\therefore$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ Converges.

$y = 1/x^2$



More Generally... The Integral Test

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$$

$a_n = f(n)$ and f is eventually nonincreasing.

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx < \infty$

these could be anything,
and they do not have to be the same

Consequence - p series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

$p > 0$

Why? $f(x) = \frac{1}{x^p}$ decreasing for $x \geq 1$.

and $\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p \leq 1 \end{cases}$

Apply the Integral Test.

Example: $\sum_{n=3}^{\infty} \frac{2}{\sqrt{n}}$

$= 2 \sum_{n=3}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test $p = \frac{1}{2} \leq 1$.

Example: $\sum_{n=2}^{\infty} \frac{3}{n^4}$ $= 3 \sum_{n=2}^{\infty} \frac{1}{n^4}$ converges by the p-series test $p = 4 > 1$.

Popper 24

1. The series $\sum_{n=1}^{\infty} \frac{1}{n}$

(0) Converges

(1) Diverges

2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(0) Converges

(1) Diverges

3. $\left\{ \frac{1}{n} \right\}$

(0) Converges

(1) Diverges

4. $\left\{ \frac{1}{n^2} \right\}$

(0) Converges

(1) Diverges

Popper 24

5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(0) Converges

(1) Diverges

6. $\left\{ \frac{1}{\sqrt{n}} \right\}$

(0) Converges

(1) Diverges

Example: $$\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$$ not a p-series

Terms: $\frac{2}{n\ln(n)} \to 0$ nonnegative

Get to work!

$f(x) = \frac{2}{x\ln(x)}$, $x \geq 2$.

f decreases b/c $x\ln(x)$ increases.

Also $$\int_2^{\infty} \frac{2}{x\ln(x)}\,dx = \lim_{t\to\infty} 2\ln\left|\ln(x)\right|\Big|_2^t$$

$$= \infty$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

by the integral test.

Example: $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$ not a p-series

Terms: $0 < \dfrac{1}{n(\ln(n))^2} \to 0$

Get to work.

$f(x) = \dfrac{1}{x(\ln(x))^2}$ $x \geq 3$.

f is decreasing b/c denom is increasing.

$$\int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{1}{x(\ln(x))^2}\,dx$$

$$= \lim_{t\to\infty} \left. \frac{-1}{\ln(x)} \right|_3^t$$

$$= \lim_{t\to\infty} \left(\underbrace{\frac{-1}{\ln(t)}}_{\to 0} + \frac{1}{\ln(3)} \right)$$

$$= \frac{1}{\ln(3)}$$

$\Rightarrow$ i.e. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$ converges by the integral test.

Comparison Tests

Idea: If the terms eventually behave like or better than the terms in a known convergent series, then the series converges. If the terms eventually behave like or worse than the terms in a known divergent series then the series diverges.

strict Comparison Test

Suppose $\boxed{0 \le a_n \le b_n}$ for n sufficiently large.

If $\sum_{n=1}^{\infty} b_n$ converges then $\boxed{\sum_{n=1}^{\infty} a_n}$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\boxed{\sum_{n=1}^{\infty} b_n}$ diverges.

Limit Comparison Test "Looks like" Test

(idea... Stated more precisely next time.)

Suppose $0 \le a_n$, $0 \le b_n$ for n sufficiently large.

If $\boxed{a_n \text{ behaves like } b_n}$ (in the limit as $n \to \infty$)

then $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Informal

made precise next time.

Example: $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

$\frac{1}{n^2+1} \sim \frac{1}{n^2}$

for n large.

and $\sum \frac{1}{n^2}$ converges.

$\therefore$ by LCT, $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

Example: $\sum_{n=3}^{\infty} \frac{2n^2+3n+1}{3n^4+5n+6}$

$\frac{2n^2+3n+1}{3n^4+5n+6} \sim \frac{2}{3n^2}$

for n large.

and $\sum \frac{2}{3n^2} = \frac{2}{3} \sum \frac{1}{n^2}$

converges.

$\therefore$ by LCT

$\sum_{n=3}^{\infty} \frac{2n^2+3n+1}{3n^4+5n+6}$ converges

Example: $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$

$\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6} \sim \frac{1}{2n}$ for n large

AND $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$

diverges.

$\therefore$ $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$ diverges by LCT.