Review: **Infinite Series...**

sums

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

terms

Could start anywhere.

Relation to the sequence of partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

sum of terms 1 through N

$$= \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

Review:

Terms

$$\sum_{n=1}^{\infty} a_n$$

Question: What if the terms of a series do not go to zero?

Answer: **(divergence test)** **The series diverges.**

Review:

Flow Chart

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} a_n = 0 \ ??$$

No

Series Diverges!!

Yes

Get to Work!!

Recall:

We can typically only determine whether an infinite series (sum) converges or diverges. Quite often, we cannot find the actual sum.

Some exceptions:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$$

Last time, we used PFD to show this sum is 1/3.

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

see the video from last time

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}}$$

$$= \frac{3}{2}$$

New:

Geometric Series: Suppose r is a given real number.

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \text{ if } r \neq 1.$$

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}$$

Why? See the previous video

More Generally...

$$\sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r} \quad \text{if} \quad |r| < 1$$

(diverges if $|r| \geq 1$).

Review: Infinite Series With Nonnegative Terms

$$\sum_{n=1}^{\infty} a_n, \quad \text{with } a_n \geq 0 \text{ for all } n.$$

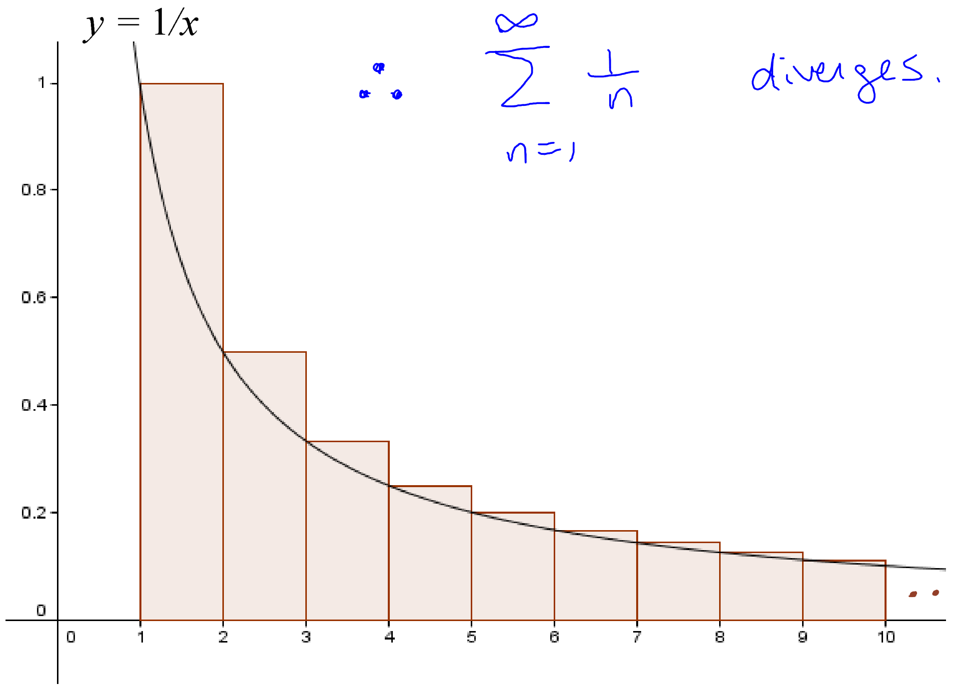
Note: In this case, what can we always say the sequence of partial sums is nondecreasing (typically, increasing).

And... a sequence that is increasing and bounded above has a limit.

Therefore, if $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ for all n, and the sum is bounded above, then it converges. Also, if the sum is not bounded above, then it diverges.

Recall: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} \geq \int_1^{\infty}\frac{1}{x}\,dx = \infty$

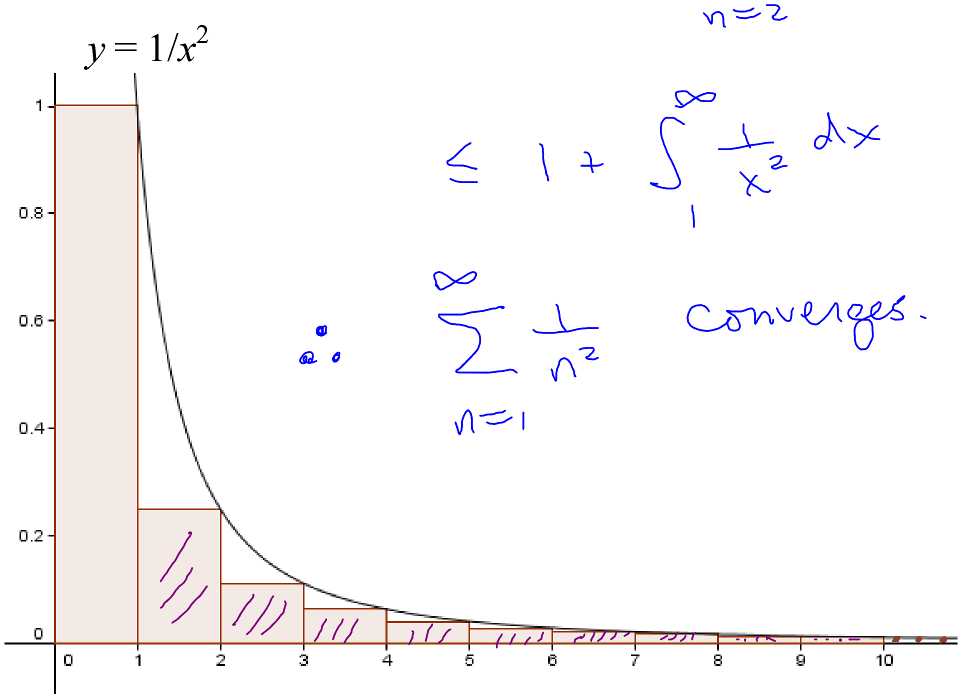
$\therefore \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.



Recall: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$

$\le 1 + \int_1^{\infty} \frac{1}{x^2}\,dx < \infty$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.



More Generally... The Integral Test

$$\sum_{n=1}^{\infty} a_n , \quad a_n \geq 0$$

$a_n = f(n)$ and f is eventually nonincreasing.

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\,dx < \infty$$

these could be anything,
and they do not have to be the same

Consequence - p series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

$p > 0$

decreasing

Why? Terms ≥ 0. Set $f(x) = \frac{1}{x^p}$.

We know $\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{finite} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$

Example: $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n}} = 2 \sum_{n=3}^{\infty} \frac{1}{n^{1/2}}$

p series with $p = \frac{1}{2} \leq 1$

$\Rightarrow$ series diverges

Example: $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n^4} = 3 \sum_{n=2}^{\infty} \frac{1}{n^4}$

p series with $p = 4 > 1$

$\Rightarrow$ series converges.

Examples:

Harmonic series : p series, p=1 .

1. The series $\sum_{n=1}^{\infty} \frac{1}{n}$

(0) Converges

(1) Diverges

p series

p=2 > 1

2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(0) Converges

(1) Diverges

3. $\left\{\frac{1}{n}\right\} \to 0$ sequence

(0) Converges

(1) Diverges

4. $\left\{\frac{1}{n^2}\right\} \to 0$ sequence

(0) Converges

(1) Diverges

Examples:

5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

p series with
$p = \frac{1}{2} \leq 1$

(0) Converges

(1) Diverges

6. $\left\{ \frac{1}{\sqrt{n}} \right\} \to 0$

sequence

(0) Converges

(1) Diverges

Example: $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$ Not a p-series.

Terms: $\dfrac{2}{n\ln(n)} \to 0$

$\dfrac{2}{n\ln(n)} \geq 0$

$f(x) = \dfrac{2}{x\ln(x)}$ $x \geq 2$

is decreasing. (b/c the numerator is fixed and the denominator is increasing.

$$\int_2^{\infty} \frac{2}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{2}{x\ln(x)}\,dx = \lim_{t\to\infty} 2\ln(|\ln(x)|)\Big|_2^{t}$$

$$= \lim_{t\to\infty} 2\left[\underbrace{\ln(\ln(t))}_{\to\infty} - \ln(\ln(2))\right]$$

$= \infty$

$\therefore$ by the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$ diverges.

Example: $$\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$$ not a p-series

Terms: $\frac{1}{n(\ln(n))^2} \to 0$

Terms are ≥ 0.

$f(x) = \frac{1}{x(\ln(x))^2}$, $x \geq 3$

$f(x)$ is decreasing. the numerator is fixed and the denominator is increasing.

$$\int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{x(\ln(x))^2}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{-1}{\ln(x)}\right)\Bigg|_3^t$$

$$= \lim_{t\to\infty} \left(\underbrace{\frac{-1}{\ln(t)}}_{\to 0} + \frac{1}{\ln(3)}\right) = \frac{1}{\ln(3)}$$

$\therefore$ The series converges from the integral test.

Comparison Tests

Idea: If the terms eventually behave like or better than the terms in a known convergent series, then the series converges. If the terms eventually behave like or worse than the terms in a known divergent series then the series diverges.

strict Comparison Test

Suppose $\boxed{0 \le a_n \le b_n}$ for n sufficiently large.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test $\leftarrow$ "Looks like"

(idea... Stated more precisely next time.)

Suppose $\boxed{0 \le a_n,\ 0 \le b_n}$ for n sufficiently large.

If $\boxed{a_n \text{ behaves like } b_n}$ (in the limit as $n \to \infty$)

then $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

ex. $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+3n^2+5}$

Terms: $\frac{2n+1}{n^3+3n^2+5} \geq 0$

behave like $\frac{2}{n^2}$ for n large.

AND $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

converges by the p-series test.

$\therefore$ by the LCT, our series converges.

Example: $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

Strict Comp. Test:

$0 \le \frac{1}{n^2+1} \le \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\therefore$ by CT our series converges.

LCT Terms: $\frac{1}{n^2+1} \geq 0$

behave like $\frac{1}{n^2}$ for n large.

AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (convergent p-series).

$\therefore$ by LCT our series converges

Example: $\sum_{n=3}^{\infty} \frac{2n^2+3n+1}{3n^4+5n+6}$

Terms: $\frac{2n^2+3n+1}{3n^4+5n+6} \geq 0$. Also,

$\frac{2n^2+3n+1}{3n^4+5n+6}$ behave like $\frac{2}{3n^2}$ for large n.

AND $\sum_{n=1}^{\infty} \frac{2}{3n^2} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test.

$\therefore$ by LCT our series converges.

Example: $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$

Terms: $\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6} \geq 0$

$\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$ behave like $\frac{1}{2n}$ for large n.

AND $\sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p series test)

$\therefore$ our series diverges.