Info

An Additional "Series" Video is Posted

7	8 EMCF32 due at 9am Blank Slides: page 4-per video notes, video Homework 11 due in lab/workshop	9 Series Video	10 EMCF33 due at 9am Homework 12 posted	11	12 EMCF34 due at 9am Quiz in lab/workshop	13 Quiz 11 closes (10.6-10.7)
14	15 EMCF35 due at 9am Homework 12 due in lab/workshop	16	17 EMCF36 due at 9am Homework 13 posted	18	19 EMCF37 due at 9am Quiz in lab/workshop	20 Quiz 12 closes (11.1-11.4)

Test 4

Comparison Tests

Idea: If the terms eventually behave like or better than the terms in a known convergent series, then the series converges. If the terms eventually behave like or worse than the terms in a known divergent series then the series diverges.

strict Comparison Test

Suppose $0 \le a_n \le b_n$ for n sufficiently large.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test ← "Looks like"

(idea... Stated more precisely below.)

Suppose $\boxed{0 \le a_n,\ 0 \le b_n}$ for n sufficiently large.

If a_n $\boxed{\text{behaves like}}$ b_n (in the limit as $n \to \infty$)

then $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \text{where} \quad 0 < L < \infty}$$

$a_n \approx L b_n$ for large n.

$0 \le a_n, \quad 0 \le b_n$

What if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$?

for large n.

eventually $0 \le a_n \le b_n$ ← more

If $\sum b_n$ converges then $\sum a_n$ converges.

If $\sum a_n$ diverges then $\sum b_n$ diverges

What if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$?

eventually $0 \le b_n \le a_n$ ← more

If $\sum a_n$ converges then $\sum b_n$ converges.

If $\sum b_n$ diverges then $\sum a_n$ diverges.

Example: $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

Terms: $\frac{1}{n^2+1} \geq 0$. $\frac{1}{n^2+1} \to 0$

(Get to work!)

3 attempts: 1

(strict) CT – $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$

AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

$\therefore$ by the C.T. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

2

LCT – $\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

$(0 < 1 < \infty)$

AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

$\therefore$ by the CT, our series converts.

3 $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\frac{1}{n^2+1} \geq 0$

Integral Test: $f(x) = \frac{1}{x^2+1}$, $x \geq 1$

$f(n) = \frac{1}{n^2+1}$

f is decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \arctan(x) \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\overset{\pi/2}{\arctan(t)} - \frac{\pi}{4}\right)$$

$$= \pi/4 < \infty$$

$\therefore$ by the integral test, the series converges.

Example: $\sum_{n=3}^{\infty} \frac{2n^2+3n+1}{3n^4+5n+6}$

Terms: $\frac{2n^2+3n+1}{3n^4+5n+6} \geq 0$

$\frac{2n^2+3n+1}{3n^4+5n+6} \to 0$

(Get to work)

$$\lim_{n\to\infty} \frac{\frac{2n^2+3n+1}{3n^4+5n+6}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^4+3n^3+n^2}{3n^4+5n+6} = \frac{2}{3}$$

$(0 < \frac{2}{3} < \infty)$

AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (conv. p-series) converges. $\therefore$ From the LCT our series converges.

Example: $\sum_{n=2}^{\infty} \frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$

Terms! $\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6} \geq 0$ $\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6} \to 0$.

(do some more work)

$$\lim_{n\to\infty} \frac{\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}}{\frac{1}{n}} = \ldots = \frac{1}{2}$$

$(0 < \frac{1}{2} < \infty.)$

AND $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series test.

$\therefore$ by the LCT, our series diverges.

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1. $\sum_{n=1}^{\infty} \frac{2n+1}{3n^3+5n}$

(0) Converges

(1) Diverges

2. $\sum_{n=2}^{\infty} \frac{2n+1}{3n^2+5n}$

(0) Converges

(1) Diverges

Root and Ratio Tests

(Determining Whether a Series Behaves Like a Geometric Series)

$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$

Idea: If the terms behave better than the terms in a convergent geometric series, then the series converges. If the terms behave worse than the terms in a divergent geometric series then the series diverges.

Root Test

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$$

changing powers

Suppose $\lim_{n\to\infty}(a_n)^{1/n} = r$. $\begin{cases} r<1 \text{ implies the series converges} \\ r>1 \text{ implies the series diverges} \\ r=1 \text{ gives no conclusion}\end{cases}$

$a_n \approx r^n$ for n large. (in a limiting sense)

Ratio Test

handles factorials and/or changing powers

Suppose $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = r$. $\begin{cases} r<1 \text{ implies the series converges} \\ r>1 \text{ implies the series diverges} \\ r=1 \text{ gives no conclusion}\end{cases}$

For n large $\quad a_{n+1} \approx r a_n$

$a_{n+2} \approx r a_{n+1} \approx r^2 a_n$

$\vdots$

$a_{n+k} \approx r^k a_n$

Recall: $n! << n^n$

Example: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

From our knowledge of growth rates, we strongly suspect that this series converges.

Terms ≥ 0

$(n+1)! = (n+1)n!$

Try ratio test: $\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}\, n^n}{(n+1)^{n+1}\, \cancel{n!}}$

Cancels

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty}\left(1 - \frac{1}{\boxed{n+1}}\right)^{\boxed{n}}$

$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$

$= \lim_{n\to\infty}\left[\left(1-\frac{1}{n+1}\right)^{n+1}\right]^{\frac{n}{n+1}}$

$\frac{n}{n+1} \to 1$, $\quad \to \frac{1}{e}$

$= \frac{1}{e} < 1$

$\therefore$ by the ratio test, our series converges.

Note: The root test would be tough to use here.

As another approach, (if you are clever) you can also use the comparison test to show this series converges.

Growth rates – $n^5 << 3^n$

Example: $\sum_{n=1}^{\infty} \frac{n^5}{3^n}$

From our knowledge of growth rates, we strongly suspect that this series converges.

$\frac{n^5}{3^n} \geq 0$

ratio + root both work

video

$\lim_{n\to\infty}\left(\frac{n^5}{3^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{5/n}}{3}$

$n^{5/n} \to 1$

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1}$$

$= \frac{1}{3} < 1$

$\therefore$ from the root test, our series converges.

Note: You can also use the ratio test to show this series converges, and if you are clever, you can use the comparison test to show the series converges.

Popper 25

3. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(0) Converges

(1) Diverges

4. $\sum_{k=1}^{\infty} \frac{k^2}{e^k}$

(0) Converges

(1) Diverges

Review

Convergence Tests for Series

1. Geometric Series
2. Divergence Test

Convergence Tests for Series with Nonnegative Entries

1. Comparison Test
2. Limit Comparison Test
3. Integral Test
4. p-series Test
5. Root Test
6. Ratio Test