Info

An Additional Series Video is Posted
Comparison Tests

**Idea:** If the terms eventually behave like or better than the terms in a known convergent series, then the series converges. If the terms eventually behave like or worse than the terms in a known divergent series then the series diverges.

**strict Comparison Test**

Suppose $0 \leq a_n \leq b_n$ for $n$ sufficiently large.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.
Limit Comparison Test

(idea... Stated more precisely below.)

Suppose \(0 \leq a_n, 0 \leq b_n\) for \(n\) sufficiently large.

If \(a_n\) behaves like \(b_n\) (in the limit as \(n \to \infty\))

then \(\sum_{n=1}^{\infty} b_n\) converges if and only if \(\sum_{n=1}^{\infty} a_n\) converges.

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L \quad \text{where} \quad 0 < L < \infty
\]
What if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) ?

\( a_n \) behaves better than \( b_n \). \( \therefore \) If \( \sum b_n \) converges, then \( \sum a_n \) converges.

What if \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) ?

If \( \sum b_n \) diverges, then \( \sum a_n \) diverges.
Example: \[ \sum_{n=2}^{\infty} \frac{1}{n^2 + 1} \]

The terms in \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) behave like the terms in \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), so is a convergent \( p \)-series.

\[ \lim_{n \to \infty} \frac{1/n^2}{1/n^2} = \lim_{n \to \infty} \frac{n^2}{n^2} = 1 \]

Also, \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is by the LCT.

Our series converges.

Example: \[ \sum_{n=3}^{\infty} \frac{2n^2 + 3n + 1}{3n^4 + 5n + 6} \]

Terms: \[ \frac{2n^2 + 3n + 1}{3n^4 + 5n + 6} \approx 0 \]

and \( \lim_{n \to \infty} \frac{2n^2 + 3n + 1}{3n^4 + 5n + 6} = \frac{2}{3} \]

\( 0 < \frac{2}{3} < \infty \)

\( \sum_{n=3}^{\infty} \frac{1}{n^2} \) is a convergent \( p \)-series.

\[ \sum_{n=3}^{\infty} \frac{1}{n^2} \]

by the LCT, our series converges.

Example: \[ \sum_{n=2}^{\infty} \frac{n^2 + 2n + 1}{2n^3 + 7\sqrt{n} + 6} \]

Terms: \[ \frac{n^2 + 2n + 1}{2n^3 + 7\sqrt{n} + 6} \approx 0 \]

and \( \lim_{n \to \infty} \frac{n^2 + 2n + 1}{2n^3 + 7\sqrt{n} + 6} = \frac{1}{2} \]

\( 0 < \frac{1}{2} < \infty \)

Also, \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) is a divergent \( p \)-series.

\[ \sum_{n=2}^{\infty} \frac{1}{n^2} \]

by the LCT, our series diverges.
Root and Ratio Tests
(Determining Whether a Series Behaves Like a Geometric Series)

\[ \sum_{n=1}^{\infty} a_n, \quad a_n \geq 0 \]

Idea: If the terms behave better than the terms in a convergent geometric series, then the series converges. If the terms behave worse than the terms in a divergent geometric series then the series diverges.
Setting: \[ \sum_{n=1}^{\infty} a_n, \ a_n \geq 0 \]

**Root Test**

Changing powers

Suppose \[ \lim_{n \to \infty} (a_n)^{1/n} = r. \]

\[
\begin{align*}
&\begin{cases}
  r < 1 \text{ implies the series converges} \\
  r > 1 \text{ implies the series diverges} \\
  r = 1 \text{ gives no conclusion}
\end{cases} \\
\Rightarrow a_n \ "\text{behaves like} \ r^n \ " \text{ as } n \to \infty
\end{align*}
\]

**Ratio Test**

Changing powers + factorials.

Suppose \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r. \]

\[
\begin{align*}
&\begin{cases}
  r < 1 \text{ implies the series converges} \\
  r > 1 \text{ implies the series diverges} \\
  r = 1 \text{ gives no conclusion}
\end{cases} \\
\Rightarrow a_{n+1} \sim r a_n \sim r^2 a_{n-1} \ldots
\end{align*}
\]
\[ n! \ll n^n \]

**Example:** \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \]

Terms:
\[
\frac{n!}{n^n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n^n} = \frac{(n+1)!}{(n+1)^n} \cdot \frac{n^n}{n^n} \]

\[
\lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \left( \frac{(n+1)!}{(n+1)^n} \cdot \frac{n^n}{n^n} \right) = \lim_{n \to \infty} \left( \frac{(n+1)!}{(n+1)^n} \cdot \frac{n^n}{n^n} \right) = \frac{1}{e} < 1
\]

\[ \therefore \text{The ratio test implies our series converges.} \]

**Note:** The root test would be tough to use here.

As another approach, (if you are clever) you can also use the comparison test to show this series converges.

\[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \]

Terms:
\[
0 \leq \frac{n!}{n^n} = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n^n} \leq \frac{2}{n^2}
\]

\[
\text{for } n \geq 2
\]

\[ \sum_{n=1}^{\infty} \frac{2}{n^2} \text{ converges by the p-series test} \]

\[ \therefore \text{Our series converges by the C.T.} \]
Example:\[
\sum_{n=1}^{\infty} \frac{n^5}{3^n}
\]

Terms: \[
\frac{n^5}{3^n} \geq 0
\]

We think this converges because of our knowledge of growth rates. Let's use the root test and ratio tests to verify this.

Root test: \[
\lim_{n \to \infty} \left( \frac{n^5}{3^n} \right)^{1/n} = \lim_{n \to \infty} \frac{n}{3} = \frac{1}{3}
\]

and \( \frac{1}{3} < 1 \), \( \therefore \) the root test implies our series converges.

Note: You can also use the ratio test to show this series converges, and if you are clever, you can use the comparison test to show the series converges.

\[
\lim_{n \to \infty} \frac{(n+1)^5}{3^{n+1}} = \lim_{n \to \infty} \frac{n^5}{3^n} \cdot \frac{1}{n} = \lim_{n \to \infty} \frac{1}{3} \cdot \left( \frac{n+1}{n} \right)^5
\]

\[
= \frac{1}{3} < 1
\]

\( \therefore \) the ratio test implies our series converges.
Review

**Convergence Tests for Series**

1. Geometric Series
2. Divergence Test

**Convergence Tests for Series with Nonnegative Entries**

1. Comparison Test ✓
2. Limit Comparison Test ✓
3. Integral Test ✓
4. p-series Test ✓
5. Root Test ✓
6. Ratio Test ✓