# Info

An Additional Series Video is Posted

# Comparison Tests

**Idea:** If the terms eventually behave like or better than the terms in a known convergent series, then the series converges. If the terms eventually behave like or worse than the terms in a known divergent series then the series diverges.

## *strict* Comparison Test

Suppose $0 \le a_n \le b_n$ for $n$ sufficiently large.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

## *Limit* Comparison Test

(idea... Stated more precisely below.)

Suppose $0 \le a_n$, $0 \le b_n$ for $n$ sufficiently large.

If $a_n$ behaves like $b_n$ (in the limit as $n \to \infty$)

then $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \text{where} \quad 0 < L < \infty}$$

What if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ ?

$a_n$ go to zero faster than $b_n$

$b_n \to 0$ + extra

$a_n$ behaves better than $b_n$. $\therefore$ If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

What if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ ? (eventually the $a_n$ are much bigger than the $b_n$)

If $\sum b_n$ diverges then

$\sum a_n$ diverges.

**Example:** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2+1}$

Terms: $\frac{1}{n^2+1} \geq 0 \sim \frac{1}{n^2}$

$$\boxed{\lim_{n\to\infty}\frac{1/(n^2+1)}{1/n^2} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1}$$

Also we know $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series.

$\therefore$ by the LCT our series converges.

The terms in $\sum \frac{1}{n^2+1}$ behave like the terms in the $\sum \frac{1}{n^2}$.

**Example:** $\displaystyle\sum_{n=3}^{\infty}\frac{2n^2+3n+1}{3n^4+5n+6}$

Terms: $\frac{2n^2+3n+1}{3n^4+5n+6} \geq 0$ and $\displaystyle\lim_{n\to\infty}\frac{\frac{2n^2+3n+1}{3n^4+5n+6}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty}\frac{2n^4+3n^3+n^2}{3n^4+5n+6} = \frac{2}{3} \quad \left(0 < \frac{2}{3} < \infty\right)$$

AND $\sum_{n=3}^{\infty}\frac{1}{n^2}$ is a convergent p-series.

$\therefore$ by the LCT, our series converges.

**Example:** $\displaystyle\sum_{n=2}^{\infty}\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}$

Terms: $\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6} \geq 0$. Also $\displaystyle\lim_{n\to\infty}\frac{\frac{n^2+2n+1}{2n^3+7\sqrt{n}+6}}{\frac{1}{n}}$

$$= \lim_{n\to\infty}\frac{n^3+2n^2+n}{2n^3+7\sqrt{n}+6} = \frac{1}{2} \quad \left(0 < \frac{1}{2} < \infty\right)$$

Also, $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent p-series.

$\therefore$ by the LCT our series diverges.

# Root and Ratio Tests

or better or worse

(Determining Whether a Series Behaves Like a Geometric Series)

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$$

could start anywhere.

**Idea:** If the terms behave better than the terms in a convergent geometric series, then the series converges. If the terms behave worse than the terms in a divergent geometric series then the series diverges.

Setting: $\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$

## Root Test

changing powers

Suppose $\underbrace{\lim_{n\to\infty} (a_n)^{1/n} = r.}$ $\begin{cases} r < 1 \text{ implies the series converges} \\ r > 1 \text{ implies the series diverges} \\ r = 1 \text{ gives no conclusion} \end{cases}$

$\hookrightarrow$ $a_n$ "behaves like" $r^n$ as $n \to \infty$

## Ratio Test

changing powers + factorials.

Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$ $\begin{cases} r < 1 \text{ implies the series converges} \\ r > 1 \text{ implies the series diverges} \\ r = 1 \text{ gives no conclusion} \end{cases}$

$a_{n+1} \approx r a_n \approx r^2 a_{n-1} \cdots$

$$n! << n^n$$

Terms: $\frac{n!}{n^n} \geq 0$

**Example:** $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

From our understanding of growth, we think this will converge. Let's use the ratio test to verify this.

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\boxed{(n+1)!}\, n^n}{(n+1)^{n+1}\boxed{n!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)}\, n^n}{(n+1)^{\cancel{n+1}}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n$$

$$= \frac{1}{e} < 1$$

$\therefore$ The ratio test implies our series converges.

**Note:** The root test would be tough to use here.

As another approach, (if you are clever) you can also use the comparison test to show this series converges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Terms:

$$0 \leq \frac{n!}{n^n} = \boxed{\frac{n\cdot(n-1)\cdot(n-2)\cdot\cdots}{n\cdot n\cdot n\cdot\cdots}}^{\leq 1} \frac{\cdot\, 2\cdot 1}{\cdot\, n\cdot n} \leq \frac{2}{n^2}$$

for $n \geq 2$

AND $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by the p-series test

$\therefore$ Our series converges by the C.T.

$n^5 << 3^n$

**Example:** $\sum_{n=1}^{\infty} \frac{n^5}{3^n}$

Terms: $\frac{n^5}{3^n} \geq 0$

We think this converges because of our knowledge of growth rates. Let's use the root test and ratio tests to verify this.

root test: $\lim_{n\to\infty} \left(\frac{n^5}{3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{5/n}}{3} = \frac{1}{3}$, $n^{5/n} \to 1$

and $\frac{1}{3} < 1$. $\therefore$ the root test implies our series converges.

**Note:** You can also use the ratio test to show this series converges, and if you are clever, you can use the comparison test to show the series converges.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^5}{3^{n+1}}}{\frac{n^5}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^5\, 3^n}{3^{n+1}\, n^5}$$

$$= \lim_{n\to\infty} \frac{1}{3} \cdot \left(\frac{n+1}{n}\right)^5, \quad \frac{n+1}{n} \to 1$$

$$= \frac{1}{3} < 1$$

$\therefore$ the ratio test implies our series converges.

# Review

**Convergence Tests for Series**

1. Geometric Series

2. Divergence Test

**Convergence Tests for Series with Nonnegative Entries**

1. Comparison Test
2. Limit Comparison Test
3. Integral Test
4. p-series Test
5. Root Test
6. Ratio Test