Useful Growth Information

If $\alpha > 0$ and $a > 1$, then

$$\ln(n) << \underbrace{n^{\alpha} << a^{n}} << n! << n^{n}$$

$\sum_{n=1}^{\infty} \frac{n^{16}}{2^n}$	$n^{16} << 2^n$ for large n	Converges. (ratio or root test)
$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$	$\ln(n) << n^{1/2}$ $\Rightarrow 0 \le \frac{\ln(n)}{n^2} < \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$ for large n.	Converges. $\sum \frac{1}{n^{3/2}}$ converges $\Rightarrow$ our series conv. by C.T.
$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$	$\ln(n) << n$ but, for $n \geq 3$ $\frac{\ln(n)}{n} \geq \frac{1}{n}$	Diverges by C.T. $\sum \frac{1}{n}$ diverges.
$\sum_{n=1}^{\infty} \frac{3^n}{n!}$	$3^n << n!$	Converges (ratio test)

New

The series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if

$$\sum_{k=1}^{\infty} |a_k| \quad \text{converges.}$$

$$\boxed{\text{If } \sum_{k=1}^{\infty} |a_k| \text{ converges, then } \sum_{k=1}^{\infty} a_k \text{ converges.}}$$

(the converse is not true)

The series $\sum_{n=1}^{\infty} a_n$ converges conditionally if and only if

the series converges, but the series does not converge absolutely.

Crummy convergence.

Conditionally convergent series are not rearrangement invariant.

Important: If a series converges absolutely, then it is rearrangement invariant. That is, the terms in the series can be added in any order, and the same value will always occur.

New

Incredible Fact: If a series is conditionally convergent, and you pick your favorite real number L, then you can rearrange the terms in the series so that the sum of the series is L. On the other hand, if a series is absolutely convergent, then the sum is the same regardless of the order in which the terms are added.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

I will post a video this afternoon showing how the terms can be rearranged "add up to" any predetermined value.

Q: Does this conv. absolutely?

A: Check $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

This diverges.

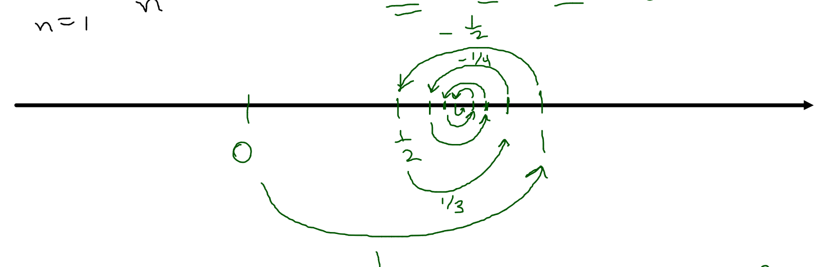
$\therefore$ our series does not converge absolutely.

Q: Does it converge conditionally.

Yes. Let's see why.

We know $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{N\to\infty} \boxed{\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}}$

Note: $\displaystyle\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{N+1}}{N}$



Convergence (Telescoping effect)

Converges.

The structure in the preceding example occurs so often that the corresponding series have a special name:

An alternating series is one that can be written in the form

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \text{or} \quad \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ with } a_k \geq 0.$$

Could be anything

Alternating Series Test: Suppose $a_k \geq 0$ for all k, $\{a_k\}$ is a decreasing sequence, and $a_k \to 0$ as $k \to \infty$. Then both

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \text{and} \quad \sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{converge.}$$

Question: How close is S_N to the full sum?

a_N

Example: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$

Determine whether the series converges absolutely or conditionally.

Absolutely? Check $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{2k+1}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+1}}$

diverges by LCT with $\displaystyle\sum \frac{1}{\sqrt{k}}$ divergent p-series.

$\therefore$ our series does not converge absolutely.

Conditionally? $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$ is an alternating series

$$\parallel$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{1}{\sqrt{2k+1}}}_{= a_k}$$

Note: $\{a_k\}$ is decreasing.

Also, $a_k \to 0$ as $k \to \infty$.

$\therefore$ by the A.S.T., $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$ converges.

$\therefore$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$ converges conditionally.

Reminder: This is crummy convergence. In fact, it is possible to rearrange the terms in this series so that they add up to any value we like.

Example: $\sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{k^3}$

Determine whether the series converges absolutely or conditionally.

Absolute Convergence?

Check $\sum_{k=1}^{\infty} \left| \frac{(-1)^k (k+1)}{k^3} \right| = \sum_{k=1}^{\infty} \frac{k+1}{k^3}$

Try a LCT with $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{k+1}{k^3}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^3+k^2}{k^3} = 1$$

Note: $0 < 1 < \infty$

Also $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a conv. p-series $p=2>1$.

$\therefore$ by the LCT, $\sum \frac{k+1}{k^3}$ converges, and this implies our series converges absolutely.

Exercises from 11.4: Absolutely or conditionally?

5. $\sum (-1)^k \frac{\ln k}{k}$. ABS × COND ✓ AST

6. $\sum (-1)^k \frac{k}{\ln k}$. $\ln(k) << k$ Terms do not go to zero. $\Rightarrow$ diverges.

7. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k!} \right)$. All terms positive Diverges

8. $\sum \frac{k^3}{2^k}$. All are positive $k^3 << 2^k$ Converges. ratio or root

9. $\sum (-1)^k \frac{1}{2k+1}$. ABS × COND ✓ AST

10. $\sum (-1)^k \frac{(k!)^2}{(2k)!}$. See next page

11. $\sum \frac{k!}{(-2)^k}$. $2^k << k!$ Terms do not go to zero. Diverges.

12. $\sum \sin \left(\frac{k\pi}{4} \right)$. Terms do not go to zero. Diverges.

13. $\sum (-1)^k \left(\sqrt{k+1} - \sqrt{k} \right)$.

14. $\sum (-1)^k \frac{k}{k^2+1}$.

15. $\sum \sin \left(\frac{\pi}{4k^2} \right)$.

16. $\sum \frac{(-1)^k}{\sqrt{k(k+1)}}$.

17. $\sum (-1)^k \frac{k}{2^k}$.

18. $\sum \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$

19. $\sum \frac{(-1)^k}{k - 2\sqrt{k}}$.

20. $\sum (-1)^k \frac{k+2}{k^2+k}$.

$$\sum (-1)^k \frac{(k!)^2}{(2k)!}$$

Abs? Check $\sum \left| (-1)^k \frac{(k!)^2}{(2k)!} \right| = \sum \frac{(k!)^2}{(2k)!}$

screaming for ratio test.

$$\lim_{k\to\infty} \frac{\frac{((k+1)!)^2}{(2(k+1))!}}{\frac{(k!)^2}{(2k)!}} = \lim_{k\to\infty} \frac{((k+1)!)^2 (2k)!}{(k!)^2 (2(k+1))!}$$

$$= \lim_{k\to\infty} \frac{\overset{(k+1)}{\cancel{(k+1)!}}\ \overset{(k+1)}{\cancel{(k+1)!}}\ \cancel{(2k)!}}{\cancel{k!}\ \cancel{k!}\ (2k+2)(2k+1)\cancel{(2k)!}}$$

$$= \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2}$$

$$= \frac{1}{4} < 1$$

$\therefore$ by the ratio test, $\sum \frac{(k!)^2}{(2k)!}$ converges.

$\Rightarrow$ our series conv. absolutely.

13. $\sum(-1)^k\left(\sqrt{k+1}-\sqrt{k}\right)$. **14.** $\sum(-1)^k \frac{k}{k^2+1}$. ABS × COND ✓ AST

15. $\sum \sin\left(\frac{\pi}{4k^2}\right)$. ABS ✓ LCT $\sum \frac{1}{k^2}$. **16.** $\sum \frac{(-1)^k}{\sqrt{k(k+1)}}$. ABS × COND ✓ AST

17. $\sum(-1)^k \frac{k}{2^k}$. ABS ✓ ratio or root **18.** $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$ All terms positive. LCT with $\sum \frac{1}{k^{3/2}}$ Conv.

19. $\sum_{k=3}^{\infty} \frac{(-1)^k}{k-2\sqrt{k}}$. ABS × COND ✓ AST **20.** $\sum(-1)^k \frac{k+2}{k^2+k}$. ABS × COND ✓ AST

$$\sum (-1)^k \frac{(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{\sqrt{k+1}+\sqrt{k}}$$

$$= \sum (-1)^k \frac{k+1-k}{\sqrt{k+1}+\sqrt{k}} = \boxed{\sum (-1)^k \frac{1}{\sqrt{k+1}+\sqrt{k}}}$$

ABS conv.? Check $\sum \frac{1}{\sqrt{k+1}+\sqrt{k}}$

Diverges by LCT with $\sum \frac{1}{\sqrt{k}}$.

∴ our series does not conv. abs.

Cond. Conv?

Alternating series.

$0 \le \frac{1}{\sqrt{k+1}+\sqrt{k}}$ decreasing for $k \ge 1$

and $\frac{1}{\sqrt{k+1}+\sqrt{k}} \to 0$.

∴ by the A.S.T., our series converges.

⇒ our series converges conditionally.

Exercises from 11.4:

ABS ✓ ratio

21. $\sum(-1)^k \frac{4^{k-2}}{e^k}$. diverges. terms do not go to zero

22. $\sum(-1)^k \frac{k^2}{2^k}$.

23. $\sum(-1)^k k \sin(1/k)$. diverges. Terms do not go to zero.

24. $\sum(-1)^{k+1} \frac{k^k}{k!}$. Diverges

$k! << k^k$ terms $\not\to 0$

25. $\sum(-1)^k k e^{-k}$. ABS ✓ ratio

26. $\sum \frac{\cos \pi k}{k}$. $= \sum \frac{(-1)^k}{k}$ ABS × COND ✓ AST

27. $\sum(-1)^k \frac{\cos \pi k}{k}$. $= \sum \frac{1}{k}$ diverges

28. $\sum \frac{\sin(\pi k/2)}{k\sqrt{k}}$.

$\left|\sin\left(\frac{\pi k}{2}\right)\right| \leq 1$

ABS ✓ C.T. with $\sum \frac{1}{k^{3/2}}$

29. $\sum \frac{\sin(\pi k/4)}{k^2}$. ABS ✓ CT. with $\sum \frac{1}{k^2}$

Note: $\left|\sin\left(\frac{\pi k}{4}\right)\right| \leq 1$

Exercises from Chapter 11 Highlights:

49. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}$ ABS ✗ COND ✓ AST

50. $\displaystyle\sum_{k=0}^{\infty} \frac{k+1}{3^k}$ Terms all pos. Converges. ratio

51. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(100)^k}{k!}$ ABS ✓ ratio

52. $\displaystyle\sum_{k=0}^{\infty} \frac{k+\cos k}{k^3+1}$ Note $|\cos(k)| \le 1$ Terms > 0 for $k \ge 1$. Conv. by LCT with $\sum \frac{1}{k^2}$

53. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(k+1)(k+2)}}$ ABS ✗ COND ✓ AST

54. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^k = \sum_{k=1}^{\infty} \frac{k}{(4/3)^k}$ Terms all positive conv. by ratio test.

55. $\displaystyle\sum_{k=0}^{\infty} \frac{k^e}{e^k}$ All terms positive. Conv. by ratio test.

56. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{\ln k}{\sqrt{k}}$ ABS ✗ COND ✓ AST

57. $\displaystyle\sum_{k=0}^{\infty} \frac{(2k)!}{2^k k!}$ Tough

ratio test $\displaystyle\lim_{k\to\infty} \frac{\frac{(2(k+1))!}{2^{k+1}(k+1)!}}{\frac{(2k)!}{2^k k!}}$

$\displaystyle= \lim_{k\to\infty} \frac{(2(k+1))!\,\cancel{2^k}\,\cancel{k!}}{\cancel{2^{k+1}}\,2\,\cancel{(k+1)!}\,(k+1)\,(2k)!}$

$\displaystyle= \lim_{k\to\infty} \frac{(2k+2)(2k+1)\cancel{(2k)!}}{2(k+1)\cancel{(2k)!}}$

$\displaystyle= \lim_{k\to\infty} \frac{4k^2+6k+2}{2k+2} = \infty$

$\infty > 1$

$\Rightarrow$ diverges by ratio test.

58. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^3+1}}$

ABS ✓

LCT with $\sum \frac{1}{k^{3/2}}$

59. $\displaystyle\sum_{k=0}^{\infty} \frac{(\tan^{-1} k)^2}{1+k^2}$ All terms positive. conv. b/c $(\tan^{-1}k)^2 \le \frac{\pi^2}{4}$ LCT with $\sum \frac{1}{k^2}$ conv.

60. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k+k^4}{3^k}$ all terms > 0 Conv. by LCT with $\sum\left(\frac{2}{3}\right)^k$