Taylor Polynomial Approximations

MOST functions have graphs that \textit{locally} look like polynomials.

\textbf{Question:} Can we approximate a function if we know the function \textit{completely} at a single point?

\[ f(x) = \sin(x) \quad \text{at} \quad x = 0 \]
\[ g(x) = \cos(x) \quad \text{at} \quad x = 0 \]
\[ h(x) = e^x \quad \text{at} \quad x = 0 \]
\[ F(x) = \ln(1+x) \quad \text{at} \quad x = 0 \]
\[ C(x) = \ln(x) \quad \text{at} \quad x = 1 \]

i.e. you know the function value and all derivative values at this point.
**Goal:** Given a function \( f \), and a value \( x = a \) where we know \( f \) and its derivatives, give a polynomial that approximates \( f \).

Degree = \( n \)

See \( p(x) \) is the polynomial.

Write \( p(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \ldots + b_n(x-a)^n \)

\[ p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]

Require: \( p(a) = f(a) \), \( p'(a) = f'(a) \), \ldots, \( p^{(n)}(a) = f^{(n)}(a) \)

\[ p(a) = b_0 = f(a) \]

\[ p'(a) = b_1 = f'(a) \]

\[ p''(x) = 2b_2 + 6b_3(x-a) + \ldots + n(n-1)b_n(x-a)^{n-1} \]

\[ p''(a) = f''(a) \]

\[ p''(a) = 2b_2 = f''(a) \Rightarrow b_2 = \frac{f''(a)}{2} \]

Similarly \( b_3 = \frac{f'''(a)}{6} \)

In general, \( b_k = \frac{f^{(k)}(a)}{k!} \).

\[ \text{ex. If } n = 4 \text{ and } a = 0, \text{ then we get} \]

\[ p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{6!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \]
Some Geogebra Plots of $\sin(x)$ Versus Taylor Polynomial Approximations Centered at $x = 0$. 

1\textsuperscript{st} degree approximation

3\textsuperscript{rd} degree approximation

5\textsuperscript{th} degree approximation

7\textsuperscript{th} degree approximation
The general process...

Given \( f(x) \), a value \( x = a \), and a positive integer \( n \), find an \( n^{th} \) degree polynomial \( p_n(x) \) so that

\[
p_n(a) = f(a), \quad p_n'(a) = f'(a), \quad \ldots, \quad p_n^{(n)}(a) = f^{(n)}(a)
\]

The **Taylor polynomial** approximation of \( f \) of degree \( n \) centered at \( x = a \) is given by

\[
p_n(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

\[
f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\]
Examples:

Give the 5th degree Taylor polynomial centered at 0 for each of \( e^x \), \( \cos(x) \), \( \sin(x) \), \( \frac{1}{1-x} \) and \( \ln(x+1) \).

\[
\begin{align*}
f(x) &= e^x & f(0) &= 1 \\
f'(x) &= e^x & f'(0) &= 1 \\
f''(x) &= e^x & f''(0) &= 1 \\
f'''(x) &= e^x & f'''(0) &= 1 \\
f^{(4)}(x) &= e^x & f^{(4)}(0) &= 1 \\
f^{(5)}(x) &= e^x & f^{(5)}(0) &= 1 \\
\end{align*}
\]

\[
\begin{align*}
P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \\
\end{align*}
\]

5th degree Taylor polynomial approximation for \( \exp(x) \) centered at 0
\[ f(x) = \cos(x) \quad f(0) = 1 \]
\[ f'(x) = -\sin(x) \quad f'(0) = 0 \]
\[ f''(x) = -\cos(x) \quad f''(0) = -1 \]
\[ f'''(x) = \sin(x) \quad f'''(0) = 0 \]
\[ f^{(4)}(x) = \cos(x) \quad f^{(4)}(0) = 1 \]
\[ f^{(5)}(x) = -\sin(x) \quad f^{(5)}(0) = 0 \]

\[ P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5 \]

\[ \approx \quad P_5(x) = 1 + 0 \cdot x - \frac{x^2}{2} + 0 \cdot x^3 + \frac{x^4}{24} + 0 \cdot x^5 \]

\[ = 1 - \frac{x^2}{2} + \frac{x^4}{24} \]

5th degree Taylor polynomial approximation for \( \cos(x) \) centered at \( 0 \).
\[
\begin{align*}
  f(x) &= \sin(x) \\
  f'(x) &= \cos(x) \\
  f''(x) &= -\sin(x) \\
  f'''(x) &= -\cos(x) \\
  f^{(4)}(x) &= \sin(x) \\
  f^{(5)}(x) &= \cos(x)
\end{align*}
\]

\[
P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5
\]

\[
P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}
\]

5th degree Taylor polynomial approximation for \(\sin(x)\) centered at 0
\[
\begin{align*}
\frac{f(x)}{1-x} &= 1 \\
\frac{f'(x)}{(1-x)^2} &= 1 \\
\frac{f''(x)}{(1-x)^3} &= 2 \\
\frac{f'''(x)}{(1-x)^4} &= 6 \\
\frac{f^{(4)}(x)}{(1-x)^5} &= 24 \\
\frac{f^{(5)}(x)}{(1-x)^6} &= 120
\end{align*}
\]

\[P_5(x) = \frac{f(0)}{1} + \frac{f'(0)}{2} x + \frac{f''(0)}{6} x^2 + \frac{f'''(0)}{24} x^3 + \frac{f^{(4)}(0)}{120} x^4 + \frac{f^{(5)}(0)}{720} x^5\]

\[= 1 + x + x^2 + x^3 + x^4 + x^5\]

5th degree Taylor polynomial approximation for \(\frac{1}{1-x}\) centered at 0

\[
\text{Recall: } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x| < 1
\]
\[ f(x) = \ln(x + 1) \]
\[ f'(x) = \frac{1}{x + 1} \]
\[ f''(x) = \frac{-1}{(x+1)^2} \]
\[ f'''(x) = \frac{2}{(x+1)^3} \]
\[ f^{(4)}(x) = -\frac{6}{(x+1)^4} \]
\[ f^{(5)}(x) = \frac{24}{(x+1)^5} \]

\[ p_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5 \]

\[ = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \]

5th degree Taylor polynomial approximation for \( \ln(1 + x) \) centered at 0