Taylor Polynomial Approximations

MOST functions have graphs
that *locally* look like polynomials.

Question: Can we approximate a function if we know the function *completely* at a single point?

i.e. you know the function value and all derivative values at this point.

1. $f(x) = \sin(x)$ at $x = 0$
2. $g(x) = \cos(x)$ at $x = 0$
3. $h(x) = e^x$ at $x = 0$
4. $F(x) = \ln(1+x)$ at $x = 0$
5. $G(x) = \ln(x)$ at $x = 1$

$\vdots$

Goal: Given a function f, and a value $x = a$ where we know f and its derivatives, give a polynomial that approximates f.

Degree $= n$

Spse $p(x)$ is the polynomial.

Write

$$p(x) = \underset{= f(a)}{b_0} + \underset{= f'(a)}{b_1}(x-a) + \underset{= \frac{f''(a)}{2}}{b_2}(x-a)^2 + \cdots + b_n(x-a)^n$$

Require: $p(a) = f(a)$, $p'(a) = f'(a)$, $\ldots$, $p^{(n)}(a) = f^{(n)}(a)$

$p(a) = b_0 = f(a)$

$p'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots + nb_n(x-a)^{n-1}$

$p'(a) = b_1 = f'(a)$

$p''(x) = 2b_2 + 6b_3(x-a) + \cdots + n(n-1)b_n(x-a)^{n-2}$

$p''(a) = f''(a)$

$p''(a) = 2b_2 = f''(a) \Rightarrow b_2 = \dfrac{f''(a)}{2}$

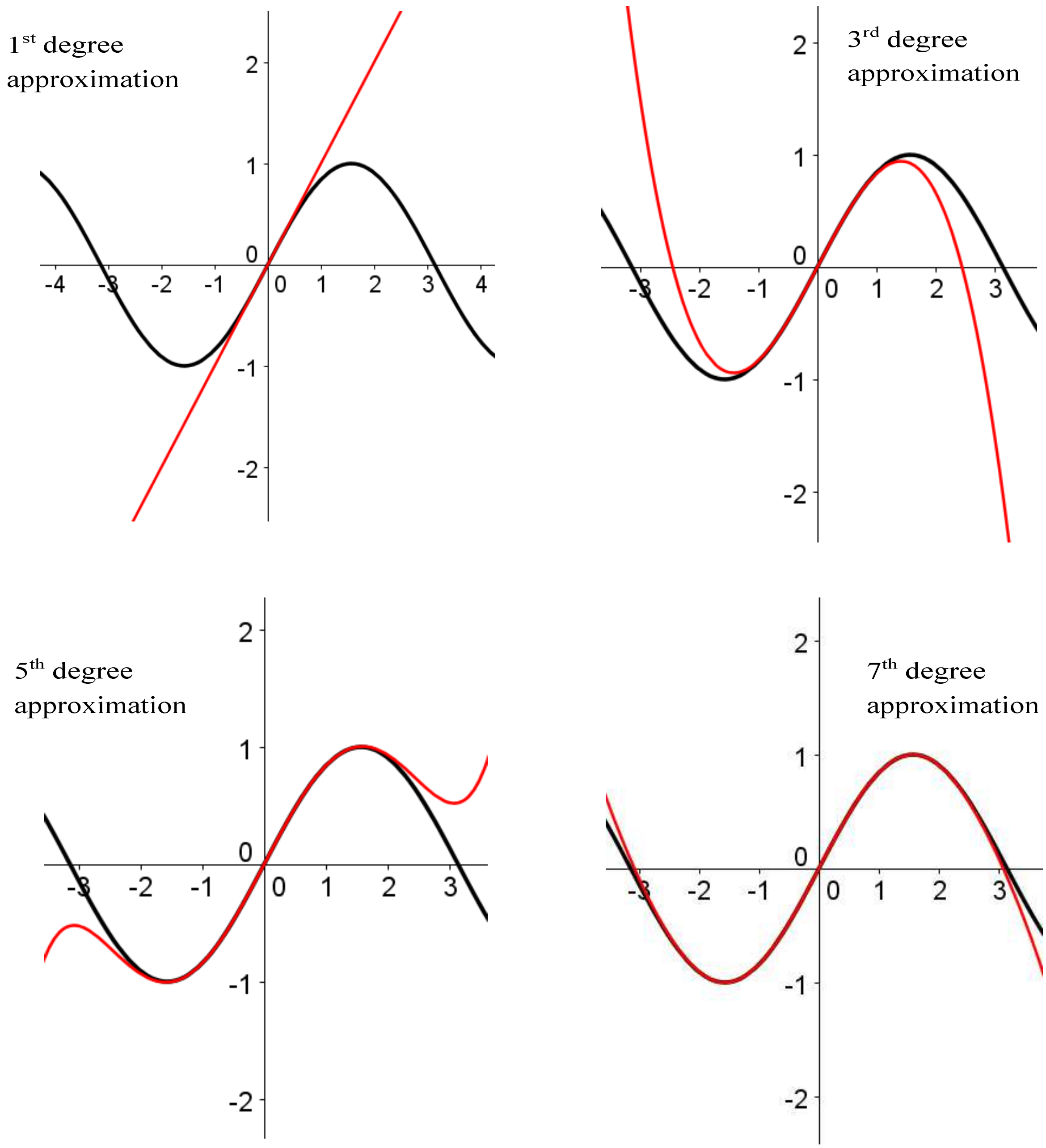
similarly $b_3 = \dfrac{f'''(a)}{6}$

In general, $b_k = \dfrac{f^{(k)}(a)}{k!}$.

ex. If $n = 4$ and $a = 0$, then we get

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

Some Geogebra Plots of sin(x) Versus *Taylor* Polynomial Approximations Centered at $x = 0$.



The general process...

Given $f(x)$, a value $x = a$, and a positive integer n, find an n^{th} degree polynomial $p_n(x)$ so that

$$p_n(a) = f(a),\ p_n'(a) = f'(a),\ \cdots,\ p_n^{(n)}(a) = f^{(n)}(a)$$

The Taylor polynomial approximation of f of degree n centered at $x = a$ is given by

$$p_n(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples:

Give the 5th degree Taylor polynomial centered at 0 for each of e^x, $\cos(x)$, $\sin(x)$, $\dfrac{1}{1-x}$ and $\ln(x+1)$.

$f(x) = e^x$ $\qquad$ $f(0) = 1$

$f^{(k)}(x) = e^x$ $\qquad$ $f^{(k)}(0) = 1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

5th degree Taylor polynomial approximation for $\exp(x)$ centered at 0

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\sin(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f'''(x) = \sin(x) \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin(x) \qquad f^{(5)}(0) = 0$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5$$

$$\therefore P_5(x) = 1 + 0x - \frac{x^2}{2} + 0x^3 + \frac{x^4}{24} + 0x^5$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

5th degree Taylor polynomial approximation for $\cos(x)$ centered at 0

$f(x) = \sin(x)$ $\qquad$ $f(0) = 0$

$f'(x) = \cos(x)$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin(x)$ $\qquad$ $f''(0) = 0$

$f'''(x) = -\cos(x)$ $\qquad$ $f'''(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\qquad$ $f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos(x)$ $\qquad$ $f^{(5)}(0) = 1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5$$

$$P_5(x) = 0 + x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120}$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120}$$

5th degree Taylor polynomial approximation for $\sin(x)$ centered at 0

$f(x) = \frac{1}{1-x}$ $\quad$ $f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$ $\quad$ $f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$ $\quad$ $f''(0) = 2$

$f'''(x) = \frac{6}{(1-x)^4}$ $\quad$ $f'''(0) = 6$

$f^{(4)}(x) = \frac{24}{(1-x)^5}$ $\quad$ $f^{(4)}(0) = 24$

$f^{(5)}(x) = \frac{120}{(1-x)^6}$ $\quad$ $f^{(5)}(0) = 120$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5$$

5th degree Taylor polynomial approximation for $1/(1-x)$ centered at 0

Recall: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ iff $|x| < 1$

$f(x) = \ln(x+1)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{x+1}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(x+1)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(x+1)^3}$ $\qquad f'''(0) = 2$

$f^{(4)}(x) = \frac{-6}{(x+1)^4}$ $\qquad f^{(4)}(0) = -6$

$f^{(5)}(x) = \frac{24}{(x+1)^5}$ $\qquad f^{(5)}(0) = 24$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5$$

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

5th degree Taylor polynomial approximation for $\ln(1 + x)$ centered at 0