Review:

What is the formula for the n^{th} degree Taylor polynomial approximation of a function $f(x)$ centered at $x = a$?

$$\underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

Notation: $p_{n,a}(x)$

Example: Give the Taylor polynomial of degree 7 centered at 0 for each of sin(x), cos(x) and exp(x).

$f(x) = \sin(x)$

$$P_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$6! = 6 \cdot 5! = 720$

$7! = 7 \cdot 6! = 5040$

$g(x) = \cos(x)$

$$P_7(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$h(x) = e^x$

$$P_7(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}$$

Example: Give the Taylor polynomial of degree 7 centered at 0 for $f(x) = e^{2x}$.

$f(0) = 1$

$f'(x) = 2e^{2x}$ $\qquad$ $f'(0) = 2$

$f''(x) = 4e^{2x}$ $\qquad$ $f''(0) = 4$

$\vdots$ $\qquad$ $\vdots$

$f^{(k)}(x) = 2^k e^{2x}$ $\qquad$ $f^{(k)}(0) = 2^k$

$$\therefore\quad P_7(x) = 1 + 2x + \frac{2^2}{2}x^2 + \frac{2^3}{6}x^3 + \frac{2^4}{24}x^4 + \frac{2^5}{120}x^5 + \frac{2^6}{720}x^6 + \frac{2^7}{5040}x^4$$

simplify ...

Note: $P_7(x) = 1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} + \cdots + \frac{(2x)^7}{5040}$

7th degree Taylor polynomial approximation for e^{2x} centered at 0

Recall: For $f(u) = e^u$, we have

$$P_7(u) = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \cdots + \frac{u^7}{5040}$$

Example: Give the Taylor polynomial of degree 4 centered at 1 for $f(x) = \ln(x)$.

$f(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4}$ $\quad f^{(4)}(1) = -6$

$$P_{4,1}(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

Example: Give the Taylor polynomial of degree 4 centered at 0 for $f(x) = x\sin(x) + 3x^2 - 2x + 1$.

$f(0) = 1$

$f'(x) = x\cos(x) + \sin(x) + 6x - 2$ $\qquad f'(0) = -2$

$f''(x) = -x\sin(x) + 2\cos(x) + 6$ $\qquad f''(0) = 8$

$f'''(x) = -x\cos(x) - 3\sin(x)$ $\qquad f'''(0) = 0$

$f^{(4)}(x) = x\sin(x) - 4\cos(x)$ $\qquad f^{(4)}(0) = -4$

$$P_4(x) = 1 - 2x + \frac{8}{2}x^2 + \frac{0}{6}x^3 - \frac{4}{24}x^4$$

$$\boxed{P_4(x) = 1 - 2x + 4x^2 - \frac{1}{6}x^4}$$

Observation:

$$f(x) = x\underbrace{\sin(x)}_{x - \frac{x^3}{6}\ \text{(up to 3rd degree)}} + \underbrace{3x^2 - 2x + 1}_{\text{2nd degree centered at 0.}}$$

$$x\left(x - \frac{x^3}{6}\right) + 3x^2 - 2x + 1$$

$$= x^2 - \frac{x^4}{6} + 3x^2 - 2x + 1 = \boxed{1 - 2x + 4x^2 - \frac{x^4}{6}}$$