Review:

What is the formula for the n^{th} degree Taylor polynomial approximation of a function $f(x)$ centered at $x = a$?

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Question: Which expansions should you know by heart?

$\sin(x)$, $\cos(x)$, e^x centered at $x=0$.

Also others...

New

estimating $|f(x) - P_{n,a}(x)|$

How do we $\boxed{\text{estimate the error}}$ associated with approximating at function $f(x)$ with its n^{th} degree Taylor polynomial approximation centered at $x = a$?

Note: $f(x) = P_{n,a}(x) + \underline{\underline{\text{error}}}$

$$\text{error} \Leftrightarrow e_{n,a}(x)$$

i.e. $\text{error} = e_{n,a}(x) = f(x) - P_{n,a}(x)$

Error Estimation

Let a and x be fixed values and suppose f is $n+1$ times differentiable on the interval connecting a and x. Then there is a value c between x and a so that

$$e_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$\text{i.e. } f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

i.e. $f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Typically, we work with an upper bound for this error, since c is not readily available.

$$|f(x) - p_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

$$\leq \underbrace{\frac{M}{(n+1)!}|x-a|^{n+1}}_{\text{bound for the error}}$$

$\boxed{c \text{ btwn } x \text{ and } a}$

where M is an upper bound for $|f^{(n+1)}(t)|$ with t btwn x and a.

Example: Let $f(x) = \cos(x)$. Determine the value of n so that $\boxed{p_{n,0}(x)}$ approximates $f(x)$ within 1/10 on the interval [-2,2].

possible x values are btwn -2 and 2

We Know

$$\left| f(x) - p_{n,0}(x) \right| \le \frac{M}{(n+1)!} \left| x - 0 \right|^{n+1}$$

?

where $\left| f^{(n+1)}(t) \right| \le M$

for t between x and 0.

Causes us to get M for all $-2 \le t \le 2$.

$f(x) = \cos(x)$

Note: $f^{(k)}(x) = \pm\cos(x)$ or $\pm\sin(x)$

$\Rightarrow \quad |f^{(n+1)}(t)| \leq 1$

M

$\therefore \quad |f(x) - P_{n,0}(x)| \leq \dfrac{1}{(n+1)!}|x|^{n+1}$

Force $\dfrac{1}{(n+1)!}|x|^{n+1} \leq \dfrac{1}{10}$

for $-2 \leq x \leq 2$.

To account for all of these x values,

we need $\dfrac{2^{n+1}}{(n+1)!} \leq \dfrac{1}{10}$.

n	$2^{n+1}/(n+1)!$
6	$128/5040$
5	$64/720 < \frac{1}{10}$
4	$32/120 > \frac{1}{10}$

$\boxed{n=5 \text{ works}}$

But, since $P_{5,0}(x) = P_{4,0}(x)$

for $f(x) = \cos(x)$,

$n=4$ will work.

Example: Give the smallest value of n so that the n^{th} degree Taylor polynomial approximation centered at 0 approximates $\exp(-2)$ within 10^{-1} on the interval.

(see the video) $f(x) = e^x$, $f^{(k)}(x) = e^x$

$$\left| \exp(-2) - P_{n,0}(-2) \right| \le \frac{M}{(n+1)!} \left| -2-0 \right|^{n+1} \le \frac{1}{10}$$

Force

$$M \ge \left| f^{(n+1)}(t) \right| = \left| e^t \right|$$

for t btwn -2 and 0.

$M = 1$ works.

Need $$\frac{2^{n+1}}{(n+1)!} \le \frac{1}{10}.$$

From earlier, $n = 5$ works.

Question: What does the error estimate tell us about using Taylor polynomials to rewrite a polynomial centered at $x = a$, for some fixed value of a?

Spse $f(x) = n^{\text{th}}$ degree polynomial

Then

$$|f(x) - P_{n,a}(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $M \ge |f^{(n+1)}(t)|$

where t btwn x and a.

Note: $f^{(n+1)}(t) \equiv 0$

since $f(x)$ is an n^{th} degree poly. So $M = 0$.

$$\therefore \quad f(x) = P_{n,a}(x)$$

$\uparrow$

n^{th} degree

Example: Rewrite the polynomial $f(x) = 2x^3 - 4x^2 + 5x + 2$ as an expansion in $x - 1$. i.e. centered at $x = 1$.

3rd degree

From earlier

$f'(x) = 6x^2 - 8x + 5$

$f''(x) = 12x - 8$

$f'''(x) = 12$

$f(x) = P_{3,1}(x)$

$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$

$= 5 + 3(x-1) + \frac{4}{2}(x-1)^2 + \frac{12}{6}(x-1)^3$

$= 5 + 3(x-1) + 2(x-1)^2 + 2(x-1)^3$

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1. Find the 5th degree Taylor polynomial centered at $x = 0$ for $\sin(x)$, and evaluated this polynomial at $x = 1$.

2. Find the 4th degree Taylor polynomial centered at $x = 0$ for $\cos(x)$, and evaluated this polynomial at $x = 1$.

3. Find the 4th degree Taylor polynomial centered at $x = 0$ for $\exp(x)$, and evaluated this polynomial at $x = 1$.