Review:

What is the formula for the $n^{th}$ degree Taylor polynomial approximation of a function $f(x)$ centered at $x = a$?

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$p_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Question: Which expansions should you know by heart?

- $\sin(x)$, $\cos(x)$, $e^x$
- Centered at $x = 0$.

Also others...
New

How do we estimate the error associated with approximating a function $f(x)$ with its $n^{th}$ degree Taylor polynomial approximation centered at $x = a$?

\[ f(x) = p_{n,a}(x) + \text{error} \]

\[ e_{n,a}(x) \]

\[ \text{error} = e_{n,a}(x) = f(x) - p_{n,a}(x) \]
Error Estimation

Let \( a \) and \( x \) be fixed values and suppose \( f \) is \( n+1 \) times differentiable on the interval connecting \( a \) and \( x \). Then there is a value \( c \) between \( x \) and \( a \) so that

\[
e_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}
\]

i.e. \( f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \)

i.e. \( f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \)

Typically, we work with an upper bound for this error, since \( c \) is not readily available.

\[
|f(x) - p_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|
\]

\[\leq \frac{M}{(n+1)!} |x-a|^{n+1}\]

Where \( M \) is an upper bound for \( |f^{(n+1)}(t)| \) with \( t \) between \( x \) and \( a \).
**Example:** Let \( f(x) = \cos(x) \). Determine the value of \( n \) so that \( p_{n,0}(x) \) approximates \( f(x) \) within \( 1/10 \) on the interval \([-2,2]\).

We know the possible \( x \) values are between \(-2\) and \(2\).

\[
|f(x) - p_{n,0}(x)| \leq \frac{M}{(n+1)!} |x-0|^{n+1}
\]

where \( |f^{(n+1)}(t)| \leq M \) for \( t \) between \( x \) and \( 0 \).

This causes us to get \( M \) for all \(-2 \leq t \leq 2\).
\[ f(x) = \cos(x) \]

**Note:** 
\[ f^{(n)}(x) = \pm \cos(x) \] or \[ \pm \sin(x) \]

\[ \Rightarrow \quad |f^{(n+1)}(x)| \leq M \]

\[ \therefore |f(x) - P_{n,0}(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \]

**Force** 
\[ \frac{1}{(n+1)!} |x|^{n+1} \leq \frac{1}{10} \]

for \(-2 \leq x \leq 2\).

To account for all of these \(x\) values, we need 
\[ \frac{2}{(n+1)!} \leq \frac{1}{10} \]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(2^{n+1} / (n+1)!)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>(128/5040)</td>
</tr>
<tr>
<td>5</td>
<td>(64/720 &lt; \frac{1}{10})</td>
</tr>
<tr>
<td>4</td>
<td>(32/120 &gt; \frac{1}{10})</td>
</tr>
</tbody>
</table>

\[ n = 5 \quad \text{works} \]

But, since \(P_{5,0}(x) = P_{4,0}(x)\)

\[ f(x) = \cos(x) \]

\(n = 4\) will work.
Example: Give the smallest value of $n$ so that the $n^{th}$ degree Taylor polynomial approximation centered at 0 approximates $\exp(-2)$ within $10^{-1}$ on the interval.

$(\text{see the video}) \quad f(x) = e^x, \quad f^{(k)}(x) = e^x$

$$\left| \exp(-2) - p_{n,0}(-2) \right| \leq \frac{M}{(n+1)!} \left| -2 - 0 \right|^{n+1} \leq \frac{1}{10}$$

Force

$$M \geq \left| f^{(n+1)}(t) \right| = |e^t|$$

for $t$ between $-2$ and 0.

$M = 1$ works.

Need

$$\frac{2^{n+1}}{(n+1)!} \leq \frac{1}{10}$$

From earlier, $n = 5$ works.
**Question:** What does the error estimate tell us about using Taylor polynomials to rewrite a polynomial centered at \( x = a \), for some fixed value of \( a \)?

Suppose \( f(x) = n^{th} \) degree polynomial

Then

\[
| f(x) - P_{n,a}(x) | \leq \frac{M}{(n+1)!}|x-a|^{n+1}
\]

where \( M \geq |f^{(n+1)}(t)| \)

where \( t \) is between \( x \) and \( a \).

*Note:* \( f^{(n+1)}(t) = 0 \)

since \( f(x) \) is an \( n^{th} \) degree poly. So \( M = 0 \).

\[
\Rightarrow f(x) = P_{n,a}(x)
\]

\( n^{th} \) degree
**Example:** Rewrite the polynomial \( f(x) = 2x^3 - 4x^2 + 5x + 2 \) as an expansion in \( x - 1 \). i.e. centered at \( x = 1 \).

\[
\begin{align*}
\text{From earlier} & \quad \text{3rd degree} \\
& \quad f'(x) = 6x^2 - 8x + 5 \\
& \quad f''(x) = 12x - 8 \\
& \quad f'''(x) = 12 \\
\end{align*}
\]

\[
\begin{align*}
f'(1) &= 6(1)^2 - 8(1) + 5 \\
f''(1) &= 12(1) - 8 \\
f'''(1) &= 12 \\
\end{align*}
\]

\[
\begin{align*}
f(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \\
&= 5 + 3(x-1) + \frac{4}{2}(x-1)^2 + \frac{12}{6}(x-1)^3 \\
&= 5 + 3(x-1) + 2(x-1)^2 + 2(x-1)^3 \\
\end{align*}
\]
Popper 29

1. Find the 5th degree Taylor polynomial centered at $x = 0$ for $\sin(x)$, and evaluated this polynomial at $x = 1$.

2. Find the 4th degree Taylor polynomial centered at $x = 0$ for $\cos(x)$, and evaluated this polynomial at $x = 1$.

3. Find the 4th degree Taylor polynomial centered at $x = 0$ for $\exp(x)$, and evaluated this polynomial at $x = 1$.