Taylor Series: (essentially, infinite degree Taylor polynomials)
Power Series

Definition: If $f(x)$ is defined and has derivatives of every order at $x = a$, then the Taylor series for $f(x)$ centered at $x = a$ is given by

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

...just like a Taylor polynomial, but the degree is infinite...

Example: Give the Taylor series for e^x, $\sin(x)$ and $\cos(x)$ centered at $x = 0$.

e^x: $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\sin(x)$: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$\cos(x)$: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Comment:

$\sin(x)$, $\cos(x)$ and e^x are equal to their Taylor Series centered at 0 for all x.

This is not the case for all functions.

e.g. $1/(1+x)$, $\ln(1+x)$, $\arctan(x)$ and many others.

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n, \quad |-x|<1$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x|<1.$$

$\downarrow$ diff

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}, \quad |-x^2|<1 \Leftrightarrow |x|<1$$

$$= \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad |x|<1$$

integrate

$$C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x|<1$$

Constant of integration

Note: $\arctan(x)$ at $x=0$ is $\underline{\underline{0}}$.

Subst $x=0$ here

$$0 = C + \sum_{n=0}^{\infty}(-1)^n\frac{0^{2n+1}}{2n+1} = C$$

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \quad |x|<1.$$

We will learn this

Example: Give the Taylor series for

$$f(x)=\frac{1}{1-x} \qquad f(0)=1$$

centered at $x=0$. For which values of x does this series converge?

$$f'(x)=\frac{1}{(1-x)^2}=\frac{1!}{(1-x)^2} \qquad f'(0)=1!$$

$$f''(x)=\frac{2}{(1-x)^3}=\frac{2!}{(1-x)^3} \qquad f''(0)=2!$$

$$f'''(x)=\frac{6}{(1-x)^4}=\frac{3!}{(1-x)^4} \qquad f'''(0)=3!$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(k)}(x)=\frac{k!}{(1-x)^{k+1}} \qquad \boxed{f^{(k)}(0)=k!}$$

T.S. centered at $x=0$ $\neq a$

$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}x^k$$

geometric series

Converges when $|x|<1$

Diverges for $|x|\geq 1$.

Also, we know

$$\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}, \quad |x|<1$$

A power series centered at a has the form

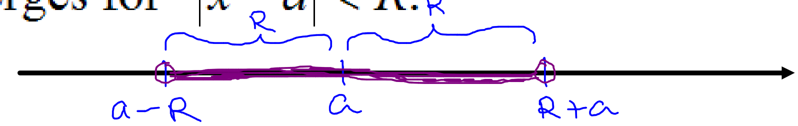
Taylor

$$\sum_{k=0}^{\infty} b_k (x-a)^k$$

Note: If this P.S. is $f(x)$, then $b_k = \frac{f^{(k)}(a)}{k!}$

The radius of convergence of a power series is the larges value of R so that the power series converges for $|x-a| < R$.

Notes:



1. Absolute convergence determines the radius of convergence.

2. If a power series is equal to a function on an interval, then the power series is the Taylor series for the function.

3. Power series can be integrated and differentiated term by term in the interior of their interval of convergence, and the power series, the derivative and the antiderivative all have the SAME radius of convergence.