Information

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A power series centered at a has the form

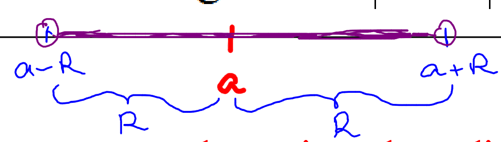
Taylor

$$\sum_{k=0}^{\infty} b_k (x-a)^k$$

If this P.S. is $f(x)$ then $b_k = \frac{f^{(k)}(a)}{k!}$

The radius of convergence of a power series is the largest value of R so that the power series converges for $|x-a| < R$.

Notes:



1. Absolute convergence determines the radius of convergence.

2. If a power series is equal to a function on an interval, then the power series is the Taylor series for the function.

3. Power series can be integrated and differentiated term by term in the interior of their interval of convergence, and the power series, the derivative and the antiderivative all have the SAME radius of convergence.

The value(s) of x where

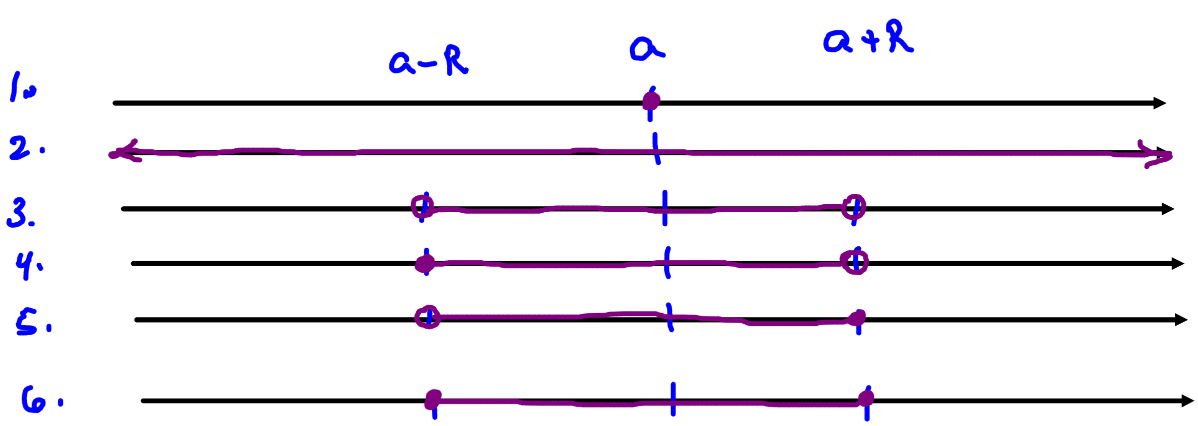
$$\sum_{k=0}^{\infty} b_k (x-a)^k$$

converges will be one of

P.S. centered at $x=a$

* **Fact:** Absolute convergence determines the radius of convergence.

1. Only $x=a$. $R=0$
2. $(-\infty, \infty)$ $R=\infty$
3. $(a-R, a+R)$ $R>0$
4. $[a-R, a+R)$
5. $(a-R, a+R]$
6. $[a-R, a+R]$.



Example: Determine the values of x where

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(x-1)^k \text{ converges,}$$

P.S. centered at $x=1$.

and give the radius of convergence.

1. Get R. Check abs. conv.

$$\sum_{k=0}^{\infty} \left| \frac{(-1)^k (x-1)^k}{k+1} \right| = \sum_{k=0}^{\infty} \frac{|x-1|^k}{k+1}$$

ratio test

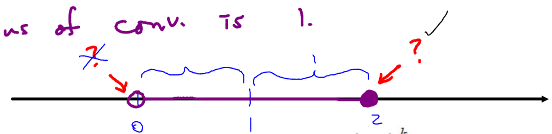
$$\lim_{k\to\infty} \frac{\frac{|x-1|^{k+1}}{k+2}}{\frac{|x-1|^k}{k+1}} = \lim_{k\to\infty} \frac{(k+1)\,|x-1|^{k+1}}{(k+2)\,|x-1|^k}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k+2}\right)|x-1| \qquad \left(\frac{k+1}{k+2} \to 1\right)$$

$$= |x-1|$$

$\therefore$ Abs. Conv. when $|x-1| < 1$. R.

Radius of conv. is 1.



$x=0$: Subst. $x=0$ into $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(x-1)^k$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(-1)^k = \sum_{k=0}^{\infty} \frac{1}{k+1}$$

Diverges

$x=2$: Subst. $x=2$ into $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(x-1)^k$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \quad \text{alternating series.}$$

$$\left(\left(\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k+1}\right.\right.$$

AST: Note: $\frac{1}{k+1} \geq 0$, $\frac{1}{k+1} \to 0$ as $k \to \infty$

AND $\frac{1}{k+1}$ decreases for $k \geq 0$
(b/c numerator is fixed and denom is increasing)

$\therefore$ Converges.

In summary: The radius of convergence is 1.

The P.S. converges for

$$0 < x \leq 2.$$

i.e. $(0, 2]$.

Important Fact: If a power series centered at $x = a$ has a radius of convergence $R > 0$, then the power series can be differentiated and integrated on $(a - R , a + R)$, and the new series will converge on $(a - R , a + R)$, and maybe at the endpoints.

Example: Find the interval and radius of convergence for

$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$. Then give the antiderivative $F(x)$ of this power series that satisfies $F(0)=2$, and find $f'(x)$. Finally, give the radius and intervals of convergence for each of $F(x)$ and $f'(x)$.

R is given by abs. conv.

$$\sum_{n=0}^{\infty}\left|\frac{x^n}{n^2+1}\right| = \sum_{n=0}^{\infty}\frac{|x|^n}{n^2+1}$$

ratio test

$$\lim_{n\to\infty}\frac{\frac{|x|^{n+1}}{(n+1)^2+1}}{\frac{|x|^n}{n^2+1}} = \lim_{n\to\infty}\frac{|x|^{n+1}(n^2+1)}{|x|^n(n^2+2n+2)}$$

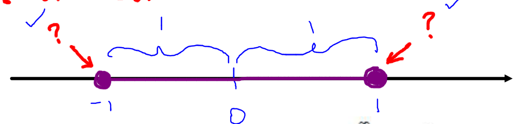
$\to 1$

$= |x|$

$\therefore$ abs. conv. for $|x|<1$.

$a=0$ R.

The radius of conv. is 1.



$x=-1$: Subst. $x=-1$ into $\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$

$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ conv. absolutely (C.T. with $\sum_{n=1}^{\infty}\frac{1}{n^2}$)

$x=1$: Subst. $x=1$ into $\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$

$\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ conv. by C.T. with $\sum_{n=1}^{\infty}\frac{1}{n^2}$

In summary: Radius of convergence is 1

AND the P.S. converges for

$-1\le x\le 1$.

i.e. on $[-1,1]$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$$

Find $F(x)$ so that $F'(x) = f(x)$ and $F(0) = 2$.

Integrate $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$ term by term.

Note: $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n$

$\therefore$ Integration (term by term) gives

$$F(x) = \left[\sum_{n=0}^{\infty} \frac{1}{n^2+1} \cdot \frac{x^{n+1}}{n+1} \right] + C$$

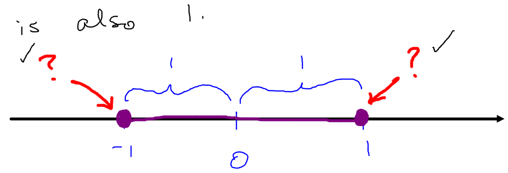
Recall: $F(0) = 2$.

$$2 = \left[\sum_{n=0}^{\infty} \frac{0^{n+1}}{(n^2+1)(n+1)} \right] + C$$

$$\Rightarrow C = 2$$

$$\therefore \quad F(x) = \left[\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n^2+1)(n+1)} \right] + 2$$

Note: Since $R=1$ for the original P.S., the radius of convergence for $F(x)$ is also 1.



$x=-1$: Subst. $x=-1$ into $F(x)$

$$\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n^2+1)(n+1)}}_{\text{conv. abs.}} + 2$$

(check it) LCT with $\sum \frac{1}{n^3}$

$x=1$: Similar. conv. at $x=1$.

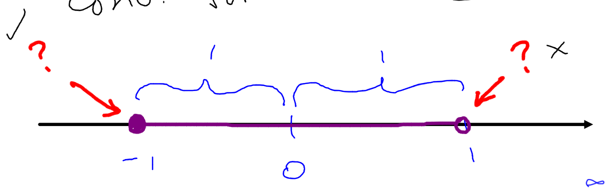
In summary, $F(x)$ has radius of convergence 1, and it converges on $[-1,1]$.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1} x^n, \quad -1 \le x \le 1$$

Diff on $(-1, 1)$ and check endpoints.

$$f'(x) = \sum_{\underline{n=1}}^{\infty} \frac{n\, x^{n-1}}{n^2+1}$$

The radius of conv. for $f'(x)$ is the same as the radius of conv. for $f(x)$. i.e. $R = 1$.



Check $x = -1$: subst. $x = -1$ into $\sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n^2+1}$

$$\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{n^2+1}$$

You can check that this is a conv. alt. series.

Check $x = 1$: subst. $x = +1$ into $\sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n^2+1}$

$$\sum_{n=1}^{\infty} \frac{n(+1)^{n-1}}{n^2+1} = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by LCT with $\sum \frac{1}{n}$

In summary: $f'(x)$ has radius of convergence 1 and it converges on

$[-1, 1)$.

Example: Let $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$. Give $f^{(9)}(0)$.

Example: Give the Taylor series centered at 0 for $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\ln(1+x)$, $\ln\left(1+x^3\right)$, and $x^2\ln\left(1+x^3\right)$.
In each case, give the radius of convergence.