Math 1432
Test 2 - Review Video

1. Determine if the following are one-to-one, if so, find $f^{-1}(x)$

a. $f(x) = x^3 + 1$ $\longmapsto$ $f'(x) = 3x^2 > 0$ except when $x=0$

b. $f(x) = 3x + 10$

c. $\boxed{f(x) = \sqrt{9 - x^2}}$ Not invertible.

$y = \sqrt{9-x^2}$

Top ½ of circle

$\Rightarrow f'(x) = 3 > 0$

$\Rightarrow f$ is 1-1.

Let's find $f^{-1}(x)$.

$y = 3x + 10$

$3x = y - 10$

$x = \frac{1}{3}(y - 10)$

Switch

$y = \frac{1}{3}(x - 10)$

$f^{-1}(x) = \frac{1}{3}(x - 10)$

$\Rightarrow f$ is increasing

$\Rightarrow f$ is 1-1.

Q: Can we find $f^{-1}(x)$?

A: $y = x^3 + 1$

$x^3 = y - 1$

$x = (y-1)^{1/3}$

switch

$y = (x-1)^{1/3}$

$\therefore f^{-1}(x) = (x-1)^{1/3}$

2. Suppose f has an inverse, $f(3)=1$ and $f'(3)=\frac{2}{7}$. Find $(f^{-1})'(1)$.

$(f^{-1})'(1) = \frac{1}{f'(a)}$ where $f(a)=1$, $a=3$

$= \frac{1}{f'(3)} = \frac{7}{2}$.

3. Suppose *f(x)* is an invertible differentiable function and the graph of *f* passes through the points (6, -1) and (-1, 2). The slope of the tangent line to the graph of *f* at $x = -1$ is 7/2. Find the equation of the tangent line to the inverse of *f* at 2.

Point: $(2, f^{-1}(2)) = (2, -1)$

Slope: $(f^{-1})'(2) = \frac{1}{f'(-1)} = \frac{1}{7/2} = \frac{2}{7}$

b/c $f(-1) = 2$.

T.L. Eq:

$y+1 = \frac{2}{7}(x-2)$.

4. Find $(f^{-1})'(a)$ if $f(x)=x^3+1$ and $a=9$

$f'(x) = 3x^2$

$(f^{-1})'(9) = \frac{1}{f'(b)}$ where $f(b)=9$

$b^3+1=9$

$b=2$

$= \frac{1}{f'(2)}$

$= \frac{1}{12}$.

5. Find the derivative:

a. $y = \ln\sqrt{e^x + 4x}$ $= \frac{1}{2}\ln(e^x + 4x)$

b. $y = \sin\left(\ln(5-x)^6\right)$

c. $y = x^2 e^{2x} + \ln e^{2x}$

d. $y = e^{x^2}\cdot\cosh(3x)$

e. $f(x) = \ln\left(\sec\sqrt{x}\right)$

$$y' = \frac{1}{2}\cdot\frac{1}{e^x+4x}\cdot(e^x+4)$$

$$= \frac{e^x+4}{2(e^x+4x)}$$

$$y = \sin\left(6\ln|5-x|\right)$$

$$y' = \cos\left(6\ln|5-x|\right)\cdot 6\cdot\frac{1}{5-x}\cdot(-1) = \frac{-6\cos(6\ln|5-x|)}{5-x}$$

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}$$

$$y = x^2 e^{2x} + 2x$$

$$y' = x^2\cdot e^{2x}\cdot 2 + e^{2x}\cdot 2x + 2$$

d. $y = e^{x^2}\cdot\cosh(3x)$ $= e^{x^2}\cdot\frac{1}{2}\left(e^{3x} + e^{-3x}\right)$

e. $f(x) = \ln\left(\sec\sqrt{x}\right)$

$$= \frac{1}{2}\left(e^{x^2+3x} + e^{x^2-3x}\right)$$

$$y' = \frac{1}{2}\left(e^{x^2+3x}\cdot(2x+3) + e^{x^2-3x}\cdot(2x-3)\right)$$

$$f'(x) = \frac{1}{\sec(\sqrt{x})}\cdot\sec(\sqrt{x})\tan(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{\tan(\sqrt{x})}{2\sqrt{x}}$$

5. Find the derivative:

f. $f(x)=\dfrac{e^{\sqrt{x}}}{x^3}$ $\longrightarrow$ $f'(x) = \dfrac{x^3 \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} - e^{\sqrt{x}} \cdot 3x^2}{x^6}$

g. $y=(\cos x)^{(x+7)}$

h. $f(x)=(3x-1)^{2x+6}$ $\longleftrightarrow$ Similar to g

i. $f(x)=\ln(5x^2)+e^{6x}+\arctan(5-2x)$

$$y = (\cos(x))^{x+7}$$

$$\ln(y) = \ln\left((\cos(x))^{x+7}\right)$$

$$\ln(y) = (x+7)\ln(\cos(x))$$

diff wrt x

$$\frac{1}{y}y' = (x+7)\frac{1}{\cos(x)}\cdot(-\sin(x)) + \ln(\cos(x))$$

$$y' = (\cos(x))^{x+7}\left[-(x+7)\tan(x) + \ln(\cos(x))\right]$$

i. $f(x)=\ln(5x^2)+e^{6x}+\arctan(5-2x)$

$$f'(x) = \frac{1}{5x^2}\cdot 10x + e^{6x}\cdot 6 + \frac{1}{1+(5-2x)^2}\cdot(-2)$$

$$= \frac{2}{x} + 6e^{6x} - \frac{2}{1+(5-2x)^2}$$

5. Find the derivative:

j. $f(x) = \log_7(3x^2)$ $\rightarrow$ $f'(x) = \frac{1}{3x^2 \ln(7)} \cdot 6x = \frac{2}{x\ln(7)}$

k. $y = 6^{-2x}$

l. $f(x) = \arctan(2x^3)$

$y' = 6^{-2x} \ln(6) \cdot (-2) = -2\ln(6) \cdot 6^{-2x}$

$f'(x) = \frac{1}{1+(2x^3)^2} \cdot 6x^2 = \frac{6x^2}{1+4x^6}$

6. Integrate:

a. $\displaystyle\int_e^{4e} \frac{1}{x}\,dx \;=\; \ln|x|\Big|_e^{4e} = \ln(4e) - \ln(e)$

$= \ln(4)$

b. $\displaystyle\int \left(\frac{\csc^2 x}{2+5\cot x} - e^{9x} \right) dx$

c. $\displaystyle\int \frac{\sinh x}{(2+\cosh x)^2}\,dx$

$$= -\frac{1}{5}\int \frac{-5\csc^2(x)}{2+5\cot(x)}\,dx - \int e^{9x}\,dx$$

$u = 2+5\cot(x)$

$du = -5\csc^2(x)\,dx$

$$= -\frac{1}{5}\ln|2+5\cot(x)| - \frac{1}{9}e^{9x} + C$$

c. $\displaystyle\int \frac{\sinh x}{(2+\cosh x)^2}\,dx \;=\; \int \underbrace{(2+\cosh(x))^{-2}}_{u^{-2}}\ \underbrace{\sinh(x)\,dx}_{du}$

$$= -(2+\cosh(x))^{-1} + C$$

$$= \frac{-1}{2+\cosh(x)} + C$$

6. Integrate:

d. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ $= 2\int e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx = 2e^{\sqrt{x}} + C$

e. $\int \frac{2}{\sqrt{x}\left(3-\sqrt{x}\right)}dx$ $= (-2)\cdot 2\int \frac{1}{(3-\sqrt{x})} \cdot \frac{(-1)}{2\sqrt{x}} dx$

$u = 3-\sqrt{x}$

$du = \frac{-1}{2\sqrt{x}}dx$

$= -4\int \frac{1}{u}du$

$= -4 \ln|3-\sqrt{x}| + C$

f. $\int \frac{x+2}{x+1}dx$

$= \int \frac{x+1+1}{x+1}dx$

$= \int \left(1 + \frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

6. Integrate:

g. $\int \frac{3x^2+3x+3}{x^2+1}dx$ $= \int \frac{3x^2+3 \quad + \quad 3x}{x^2+1} dx$

h. $\int \frac{\cos^3 x - \sin^2 x}{\cos^2 x} dx$

i. $\int \tan(3x)dx$

$= \int \left(3 + \frac{3x}{x^2+1}\right) dx$

$= 3x + \frac{3}{2}\int \frac{2x}{x^2+1} dx$

$= 3x + \frac{3}{2}\ln\left|x^2+1\right| + C$

$= \int \left(\cos(x) - \tan^2(x)\right) dx$

$\tan^2(x) = \sec^2(x) - 1$

$= \sin(x) - \int \tan^2(x)\, dx$

$= \sin(x) - \int \left(\sec^2(x) - 1\right) dx$

$= \sin(x) - \left(\tan(x) - x\right) + C$

$= \sin(x) - \tan(x) + x + C$

i. $\int \tan(3x)dx$ $= \frac{1}{3}\int \tan(3x)\, 3\, dx$

$= -\frac{1}{3}\ln\left|\cos(3x)\right| + C$

$\int \tan(u)du$

$= -\ln|\cos(u)| + C$

or

$\ln|\sec(u)| + C$

6. Integrate:

j. $\int \frac{\arctan(3x)}{1+9x^2}dx$ $= \frac{1}{3}\int \arctan(3x) \cdot \frac{1}{1+9x^2} 3\,dx$

$u = \arctan(3x)$

$du = \frac{1}{1+(3x)^2} \cdot 3\,dx$

$= \frac{1}{1+9x^2} \cdot 3\,dx$

$= \frac{1}{3}\int u\,du = \frac{1}{6}u^2 + C$

$= \frac{1}{6}\left(\arctan(3x)\right)^2 + C$

k. $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}dx$

l. $\int \cos^4 x \sin^3 x dx$

L.5 $\int \sin^4(x)\,dx$

$= \arcsin(x) \Big|_0^{\sqrt{3}/2}$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)$

$= \frac{\pi}{3} - 0 = \frac{\pi}{3}.$

1. $\int \cos^4 x \sin^3 x dx = \int \cos^4(x)\sin^2(x)\sin(x)\,dx$

L.5 $\int \sin^4(x)\,dx$

$= \int \cos^4(x)\left(1-\cos^2(x)\right)\sin(x)\,dx$

$= -\int \cos^4(x)(-\sin(x))\,dx + \int \cos^6(x)(-\sin(x))\,dx$

$= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$

$= \int \sin^4(x)\,dx$ $\qquad \sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$= \int \left(\sin^2(x)\right)^2 dx$

$= \int \left(\frac{1}{2}(1-\cos(2x))\right)^2 dx$

$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right)dx$

$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)\,dx$

$\cos^2(2x) = \frac{1}{2}\left(1+\cos(4x)\right)$

$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}\int\left(1+\cos(4x)\right)dx$

$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$

6. Integrate:

m. $\int \cos^5 x \sin^2 x dx$ $\Big] \longrightarrow$ $\int \cos^4(x)\sin^2(x)\underbrace{\cos(x)dx}$

n. $\int \cot^3 xdx$

o. $\int x\ln(2x)dx$

$$= \int (\cos^2(x))^2 \sin^2(x) \underbrace{\cos(x)dx}$$

o.5 $\int x\cos(2x)\,dx$

$$= \int (1-\sin^2(x))^2 \sin^2(x) \underbrace{\cos(x)dx}$$

o.6 $\int x\arctan(x)\,dx$

$$= \int (1-2\sin^2(x)+\sin^4(x))\sin^2(x)\underbrace{\cos(x)dx}$$

$$= \int \sin^2(x)\underbrace{\cos(x)dx} - 2\int \sin^4(x)\underbrace{\cos(x)dx} + \int \sin^6(x)\underbrace{\cos(x)dx}$$

$$= \frac{1}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + \frac{1}{7}\sin^7(x) + C$$

n. $\int \cot^3 xdx$ $= \int \cot(x)\cot^2(x)dx$

$$= \int \cot(x)\left(\csc^2(x)-1\right)dx$$

$$= \int \cot(x)\csc^2(x)dx - \int \cot(x)dx$$

$$= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C$$

o. $\int x\ln(2x)dx$

o.5 $\int x\cos(2x)\,dx$

o.6 $\int x\arctan(x)\,dx$

all parts

$$\int x\ln(2x)dx = uv - \int v\,du$$

$u=\ln(2x)$ $\quad du=\frac{1}{x}dx$

$dv=x\,dx$ $\quad v=\frac{1}{2}x^2$

$$= \frac{1}{2}x^2\ln(2x) - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2\ln(2x) - \frac{1}{4}x^2 + C$$

$$\int x\cos(5x)dx = uv - \int v\,du$$

$u=x$ $\quad du=dx$

$dv=\cos(5x)dx$ $\quad v=\frac{1}{5}\sin(5x)$

$$= \frac{x}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)dx$$

$$= \frac{x}{5}\sin(5x) + \frac{1}{25}\cos(5x) + C$$

$$\int x\arctan(x)dx = \left(\frac{1}{2}x^2+\frac{1}{2}\right)\arctan(x) - \int \frac{1}{2}dx$$

$u=\arctan(x)$ $\quad du=\frac{1}{1+x^2}dx$

$dv=x\,dx$ $\quad v=\frac{1}{2}x^2+\frac{1}{2}$

$$= \left(\frac{1}{2}x^2+\frac{1}{2}\right)\arctan(x) - \frac{1}{2}x + C$$

6. Integrate:

p. $\int 2x\sin(3x)dx$ → Like 0.5.

q. $\int \frac{5}{36+(x-1)^2}dx$ → Reminder: $\int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

r. $\int \tan^4(x)dx$

r.5 $\int \frac{x}{\sqrt{9-3x^4}}dx$

$$= 5\int \frac{1}{6^2+(x-1)^2}dx$$

$$= 5\cdot\frac{1}{6}\arctan\left(\frac{x-1}{6}\right)+C$$

$$\int \tan^4(x)dx = \int \tan^2(x)\tan^2(x)dx$$

$$= \int \tan^2(x)\left(\sec^2(x)-1\right)dx$$

$$= \int \tan^2(x)\sec^2(x)dx - \int \tan^2(x)dx$$

$$= \frac{1}{3}\tan^3(x) - \int \tan^2(x)dx$$

$$= \frac{1}{3}\tan^3(x) - \int (\sec^2(x)-1)dx$$

$$= \frac{1}{3}\tan^3(x) - \left(\tan(x)-x\right) + C$$

$$= \frac{1}{3}\tan^3(x) - \tan(x) + x + C.$$

r.5 $\int \frac{x}{\sqrt{9-3x^4}}\,dx$

Reminder: $\int \frac{1}{\sqrt{a^2-u^2}}\,du$

$= \arcsin\left(\frac{u}{a}\right) + C$

$= \frac{1}{2\sqrt{3}} \int \frac{2\sqrt{3}\,x}{\sqrt{3^2 - (\underbrace{\sqrt{3}\,x^2}_{u})^2}}\,dx$

$u = \sqrt{3}\,x^2$

$du = 2\sqrt{3}\,x\,dx$

$= \frac{1}{2\sqrt{3}} \int \frac{1}{\sqrt{3^2 - u^2}}\,du$

$= \frac{1}{2\sqrt{3}} \arcsin\left(\frac{u}{3}\right) + C$

$= \frac{1}{2\sqrt{3}} \arcsin\left(\frac{\sqrt{3}\,x^2}{3}\right) + C.$

6. Integrate:

s. $\int 2x\sec\left(4x^2\right)dx$

$u = 4x^2$

$du = 8x\,dx$

$\frac{1}{4}\int \sec(4x^2)\cdot 8x\,dx$

$= \frac{1}{4}\int \sec(u)\,du$

$= \frac{1}{4}\ln\left|\sec(u) + \tan(u)\right| + C$

$= \frac{1}{4}\ln\left|\sec(4x^2) + \tan(4x^2)\right| + C$

t. $\int \sec^4(x)dx$

t.5 $\int \tan^3(x)\sec^3(x)\,dx$

$$\int \sec^4(x)\,dx = \int \sec^2(x)\,\sec^2(x)\,dx$$

$$= \int \left(1 + \tan^2(x)\right)\sec^2(x)\,dx$$

$$= \int \sec^2(x)\,dx + \int \tan^2(x)\sec^2(x)\,dx$$

$$= \tan(x) + \frac{1}{3}\tan^3(x) + C$$

t.5 $\int \tan^3(x)\sec^3(x)\,dx = \int \tan^2(x)\sec^2(x)\,\sec(x)\tan(x)\,dx$

$$= \int \left(\sec^2(x) - 1\right)\sec^2(x)\,\sec(x)\tan(x)\,dx$$

$$= \int \sec^4(x)\,\sec(x)\tan(x)\,dx - \int \sec^2(x)\,\sec(x)\tan(x)\,dx$$

$$= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$$

7. Give the general solution for $\frac{dy}{dx} = (y+5)(x+2)$

$$\int \frac{dy}{y+5} = \int (x+2)\,dx \Rightarrow \ln|y+5| = \frac{1}{2}(x+2)^2 + C$$

8. Find the specific solution given the initial condition: $\frac{dy}{dx} = y - 2 \quad y(0) = 6$

Note: If $w = y-2$ then $\frac{dw}{dx} = \frac{dy}{dx} = y - 2 = w$

$$\Rightarrow w = w(0)e^x = (y(0)-2)e^x = 4e^x$$

$y - 2 = 4e^x$

$\boxed{y = 2 + 4e^x}$.

8.5 Solve $\frac{dy}{dt} = -3y,\ y(0) = 2.$

$$y = y(0)e^{-3t} \Rightarrow y = 2e^{-3t}$$

9. The number N of bacteria in a culture is given by $N = 200e^{kt}$. If N = 300 when t = 4 hours, find k (to the nearest tenth) and then determine approximately how long it will take for the number of bacteria to triple in size.

$N = 200e^{kt}$ $N(0) = 200.$

Note: $N(4) = 300$

$300 = 200e^{k \cdot 4} \Rightarrow \frac{3}{2} = e^{4k}$

$\ln\left(\frac{3}{2}\right) = 4k \Rightarrow k = \boxed{\frac{1}{4}\ln\left(\frac{3}{2}\right)}$

≈ 0.101366

Triple in size? Solve

$600 = 200e^{kt}$

$3 = e^{kt}$

$\ln(3) = kt$

$\boxed{\frac{4\ln(3)}{\ln(3/2)}} = t \approx 10.83805$ hrs.

10. Suppose that the population of Zeegers grows at a rate proportional to itself, doubling every 12500 years. When the Zeeger population has reached 93 percent more than their current population, they plan to invade Earth. How many years will it be before the Zeegers attack Earth?

$$Z' = kZ \Rightarrow \boxed{Z = Z(0)e^{kt}}$$

$$\boxed{Z(12{,}500) = 2Z(0)}$$

Find t so that $1.93\,Z(0) = Z(t)$.

$$2\cancel{Z(0)} = \cancel{Z(0)}e^{k\cdot 12500}$$

$$2 = e^{k\cdot 12500}$$

$$\ln(2) = k\cdot 12500 \Rightarrow \boxed{k = \frac{\ln(2)}{12500}}$$

$$1.93\,Z(0) = Z(0)e^{kt}$$

$$1.93 = e^{kt}$$

$$\ln(1.93) = kt$$

$$t = \frac{12500\ln(1.93)}{\ln(2)} \approx 11{,}857.51 \text{ yrs}$$

11. At what rate *r* of continuous compounding does a sum of money double in 15 years?

$r = k$

$$A(t) = A(0)e^{kt}$$

$$2\cancel{A(0)} = \cancel{A(0)}e^{k \cdot 15}$$

$$\ln(2) = k \cdot 15$$

$$k = \frac{\ln(2)}{15} \approx 0.04621$$

i.e. 4.621%.

12. Give the equation for the tangent and normal to the curve: $f(x)=\ln(2x-5)+e^{x-3}$ at the point (3,1).

$$\boxed{\begin{array}{l} \text{T.L. Eq: } y-1=3(x-3) \\ \text{N.L Eq: } y-1=-\frac{1}{3}(x-3) \end{array}}$$

Point: $(3,1)$

Slope: T.L.: $f'(3)=3$ N.L.: $-\frac{1}{3}$

$$f'(x)=\frac{1}{2x-5}\cdot 2+e^{x-3}$$

$$f'(3)=2+1=3$$