Test 3 Online Review
Spring 2012
Sections 8.4 - 10.3

10 questions: 3 multiple choice and 7 written

- Trigonometric Substitution
- Partitial Fraction Decomposition
- Numerical Integration
- Polar Coordinates
- Parametric Curves
- Sets, Sequences, LUB, GLB, Monotonicity and Limits

Part I - Trigonometric Substitution

$a > 0$

For integrals involving...	Use the substitution...
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$

$$\int \frac{x^2}{\sqrt{25+x^2}}\,dx = \int \frac{25\tan^2(\theta)\,5\sec^2(\theta)\,d\theta}{\sqrt{25+25\tan^2(\theta)}}$$

$x = 5\tan(\theta)$

$dx = 5\sec^2(\theta)\,d\theta$

$$= \int \frac{25\tan^2(\theta)\,\cancel{5}\sec^2(\theta)\,d\theta}{\cancel{5}\sqrt{\underbrace{1+\tan^2(\theta)}}}$$

$$= \int \frac{25\tan^2(\theta)\sec^2(\theta)}{\sqrt{1+\tan^2(\theta)}}\,d\theta$$

Note:

$1+\tan^2(\theta) = \sec^2(\theta)$

$\sqrt{\sec^2(\theta)} = \sec(\theta)$

$$= 25\int \frac{\tan^2(\theta)\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= 25\int \tan^2(\theta)\sec(\theta)\,d\theta$$

(A little harder than the exam.)

$$= 25\int (\sec^2(\theta)-1)\sec(\theta)\,d\theta$$

$$= 25\int (\sec^3(\theta)-\sec(\theta))\,d\theta$$

$\vdots$

$\boxed{\text{At the end, get the answer in terms of } x.}$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2(\theta)}\cdot 2\cos(\theta)\,d\theta$$

$x = 2\sin(\theta)$

$dx = 2\cos(\theta)\,d\theta$

$$= \int 2\underbrace{\sqrt{1-\sin^2(\theta)}}\cdot 2\cos(\theta)\,d\theta$$

Note: $1-\sin^2(\theta) = \cos^2(\theta)$

$\sqrt{\cos^2(\theta)} = \cos(\theta)$

$$= 4\int \cos^2(\theta)\,d\theta$$

$$= 4\int\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta$$

$$= 2\theta + \sin(2\theta) + C$$

(*)

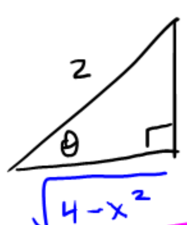
Now, give the answer in terms of x.

$x = 2\sin(\theta) \Rightarrow \boxed{\frac{x}{2} = \sin(\theta)}$

$\boxed{\arcsin\left(\frac{x}{2}\right) = \theta}$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\sin(\theta) = \frac{x}{2}$



$\Rightarrow \boxed{\cos(\theta) = \frac{1}{2}\sqrt{4-x^2}}$

$$= 2\theta + 2\sin(\theta)\cos(\theta) + C$$

?

$$= 2\arcsin\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{1}{2}\sqrt{4-x^2} + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C$$

Part II - Partial Fraction Decomposition

Setting: $\int \frac{p(x)}{q(x)}dx$, where $p(x)$ and $q(x)$ are polynomials

and the $\underbrace{\text{degree of } q(x)} > \underbrace{\text{degree of } p(x)}$

Process: 1. Do PFD on $\frac{p(x)}{q(x)}$

(factor $q(x)$ and/or identify irreducible quadratic factors)

2. Integrate the PFD.

$$\int \frac{2x-1}{x^2-3x-10}\,dx = \int \frac{2x-1}{(x-5)(x+2)}\,dx$$

degree 1

degree 2

1. Do a PFD on $\dfrac{2x-1}{(x-5)(x+2)}$

$$\frac{2x-1}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$$

This is the PFD (once you have A and B)

Mult. by $(x-5)(x+2)$

$$2x-1 = A(x+2) + B(x-5)$$

killer x: $x=5$

Subst. $x=5$

$$9 = 7A \Rightarrow \boxed{A = \frac{9}{7}}$$

killer x: $x=-2$

$$-5 = -7B \Rightarrow \boxed{B = \frac{5}{7}}$$

$$\therefore \frac{2x-1}{(x-5)(x+2)} = \frac{9/7}{x-5} + \frac{5/7}{x+2} \quad (*)$$

2. Integrate.

$$\int \frac{2x-1}{(x-5)(x+2)}\,dx = \int\left(\frac{9/7}{x-5} + \frac{5/7}{x+2}\right)dx$$

$$= \frac{9}{7}\ln|x-5| + \frac{5}{7}\ln|x+2| + C.$$

$$\int \frac{2x^2-1}{(x-1)(x^2+4)}\,dx =$$

degree = 2

degree = 3

$3 > 2$

$\Rightarrow$ no division needed.

1. Do a PFD on $\dfrac{2x^2-1}{(x-1)(x^2+4)}$

linear

irreducible quadratic

$$\frac{2x^2-1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

this is the PFD (once we know A, B, C)

mult. through by $(x-1)(x^2+4)$

$$2x^2-1 = A(x^2+4) + (Bx+C)(x-1)$$

killer x: $x=1$ subst. $x=1$.

$$1 = 5A \Rightarrow \boxed{A = \tfrac{1}{5}}$$

other x values: $x=0$ subst $x=0$.

$$-1 = \tfrac{1}{5}\cdot 4 + C(-1) \Rightarrow -\tfrac{9}{5} = -C \Rightarrow \boxed{C = \tfrac{9}{5}}$$

$x=-1$ subst $x=-1$

$$1 = \tfrac{1}{5}\cdot 5 + \left(B(-1) + \tfrac{9}{5}\right)(-2) \Rightarrow \boxed{B = \tfrac{9}{5}}$$

$$\therefore \quad \frac{2x^2-1}{(x-1)(x^2+4)} = \frac{1/5}{x-1} + \frac{\frac{9}{5}x + \frac{9}{5}}{x^2+4}$$

this is the PFD

2. Integrate

$$\int \frac{2x^2-1}{(x-1)(x^2+4)}dx = \int\left[\frac{1/5}{x-1} + \frac{\frac{9}{5}x + \frac{9}{5}}{x^2+4}\right]dx$$

$$= \frac{1}{5}\ln|x-1| + \frac{9}{5}\underbrace{\int \frac{x+1}{x^2+4}dx}_{??}$$

Break it up.

$$= \frac{1}{5}\ln|x-1| + \frac{9}{5}\left[\frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}\,dx\right]$$

$$= \frac{1}{5}\ln|x-1| + \frac{9}{10}\ln|x^2+4| + \underbrace{\frac{9}{5}\cdot\frac{1}{2}}\arctan\left(\frac{x}{2}\right) + C$$

$$= \frac{1}{5}\ln|x-1| + \frac{9}{10}\ln(x^2+4) + \frac{9}{10}\arctan\left(\frac{x}{2}\right) + C$$

$$\int \frac{2x^2-1}{(x-1)(x+2)^2}\,dx =$$

You (mainly)

1. PFD

$$\frac{2x^2-1}{\underbrace{(x-1)}\underbrace{(x+2)^2}} = \underbrace{\frac{A}{x-1}} + \underbrace{\frac{B}{x+2} + \frac{C}{(x+2)^2}}$$

Find A, B, C.

2. Integrate You!!

Trapezoid
Midpoint
Simpsons.

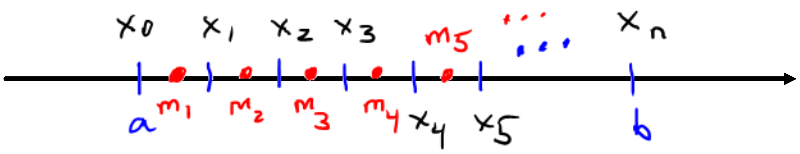
Part III - Numerical Integration

Tools for approximating $\int_a^b f(x)\,dx$

$$m_i = \frac{x_i + x_{i-1}}{2}$$

$n > 0$ (integer)

Set up:



$$T_n = \left(f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1})\right) + f(x_n)\right)\frac{b-a}{2n}$$

$$M_n = \left(f(m_1) + f(m_2) + \cdots + f(m_n)\right)\frac{b-a}{n}$$

$$S_n = \boxed{} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

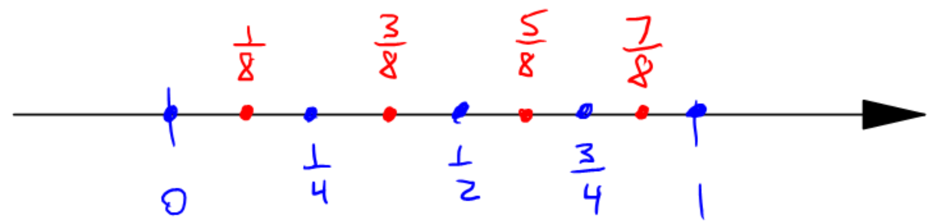
Estimate $\int_0^1 \frac{1}{x+1}\,dx$ using the trapezoid method, the midpoint method, and Simpson's method with $n=4$.

$a=0,\ b=1$

$n=4$

$n=4$

$f(x)=\frac{1}{x+1}$



$$T_4 = \left(f(0) + 2\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4})\right) + f(1)\right)\underbrace{\frac{b-a}{2n}}_{=\frac{1}{8}}$$

$$= \underbrace{\left(1 + 2\left(\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7}\right) + \tfrac{1}{2}\right)\cdot \tfrac{1}{8}}_{\text{arithmetic}}$$

(it will be a little simpler on the Test)

$f(x)=\frac{1}{x+1}$

$$M_4 = \left(f(\tfrac{1}{8}) + f(\tfrac{3}{8}) + f(\tfrac{5}{8}) + f(\tfrac{7}{8})\right)\cdot \underbrace{\frac{b-a}{n}}_{=\frac{1}{4}}$$

$$= \underbrace{\left(\tfrac{8}{9} + \tfrac{8}{11} + \tfrac{8}{13} + \tfrac{8}{15}\right)\cdot \tfrac{1}{4}}_{\text{arithmetic}}$$

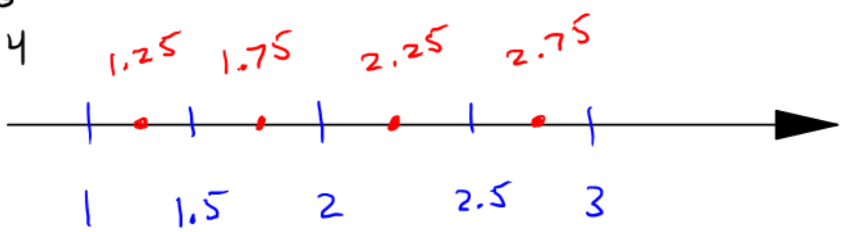
Then $S_4 = \boxed{\tfrac{1}{3}T_4 + \tfrac{2}{3}M_4}$ Do it!

Use the midpoint, trapezoid and Simpson's methods to

estimate $\int_1^3 f(x)\,dx$ with $n = 4$.

integral from 1 to 3.

$a = 1$
$b = 3$
$n = 4$



x	$f(x)$
0	1
.25	.8
.5	.7
.75	.7
1	.6
1.25	.5
1.5	.4
1.75	.3
2	.3
2.25	.2
2.5	.1
2.75	0
3	0
3.25	.1
3.5	.2
3.75	.3
4	.4

$$T_4 = \left(f(1) + 2\left(f(1.5) + f(2) + f(2.5)\right) + f(3)\right)\underbrace{\frac{b-a}{2n}}_{\frac{1}{4}}$$

$$= \left(.6 + 2(.4 + .3 + .1) + 0\right)\frac{1}{4}$$

$$= (.6 + 1.6)\frac{1}{4} = \frac{2.2}{4} = 0.55$$

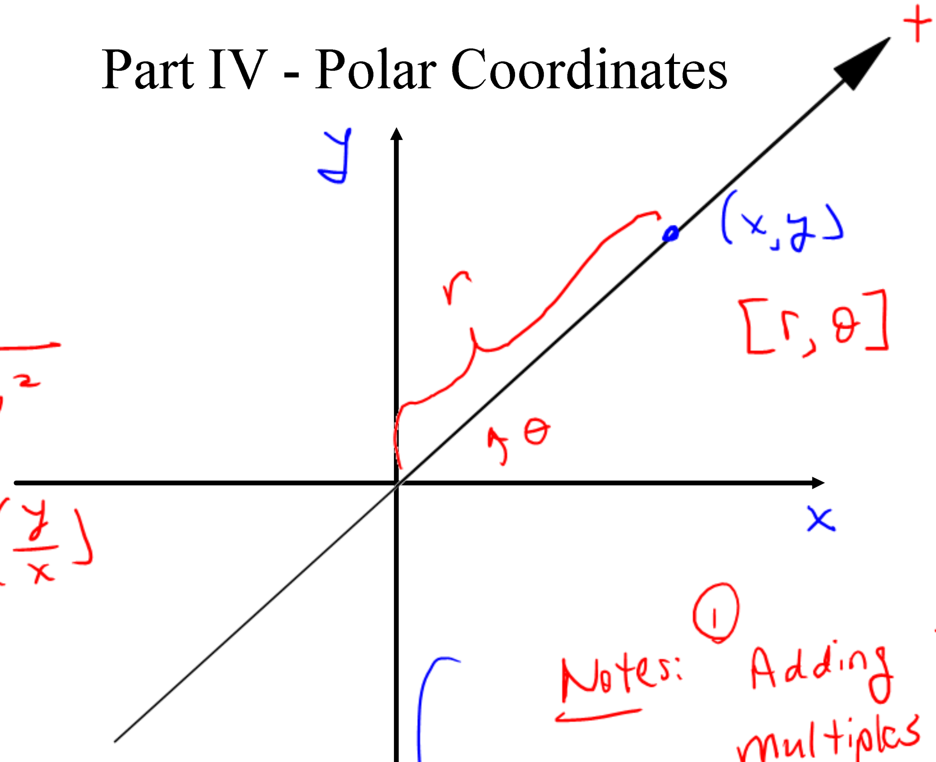
$$M_4 = \left(f(1.25) + f(1.75) + f(2.25) + f(2.75)\right)\underbrace{\frac{b-a}{n}}_{=\frac{1}{2}}$$

$$= (.5 + .3 + .2 + 0)\frac{1}{2}$$

$$= \frac{1}{2}$$

$$S_4 = \frac{1}{3}T_4 + \frac{2}{3}M_4 = \frac{0.55}{3} + \frac{1}{3} = \frac{1.55}{3} = 0.51\overline{6}$$

Part IV - Polar Coordinates



Standard

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

For Quadrants 1 and 4.

Notes: ① Adding integer multiples of 2π to θ gives other representations with r above.

② using the negative of this r and θ given by the one above plus π plus integer multiples of 2π gives others.

∞ many polar representations.

Note:

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x^2+y^2 = r^2$.

Give rectangular coordinates for the polar point $\left[-2, \frac{\pi}{3}\right]$.

x and y

r θ

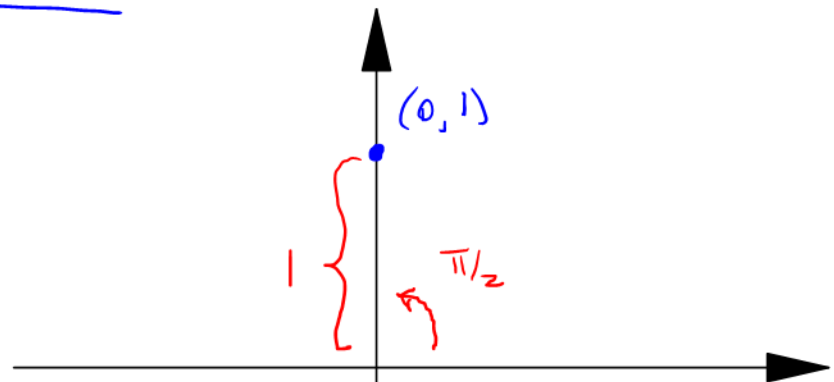
$$x = r\cos(\theta) = -2\cos(\pi/3) = -1$$

$$y = r\sin(\theta) = -2\sin(\pi/3) = -\sqrt{3}$$

$$\left(-1, -\sqrt{3}\right).$$

Points are specified in rectangular coordinates. Give all possible polar coordinates for each point.

- $(0, 1)$.
- $(-3, 0)$.
- $(2, -2)$.
- $(4\sqrt{3}, 4)$.



$[1, \pi/2]$, $[-1, \frac{3\pi}{2}]$,

$[1, \frac{\pi}{2} + 2\pi k]$, $[-1, \frac{3\pi}{2} + 2\pi k]$

$k = \pm 1, \pm 2, \ldots$ $\qquad$ $k = \pm 1, \pm 2, \ldots$

$x = r\cos(\theta)$
$y = r\sin(\theta)$

Write the equation in polar coordinates:

$y = -6 \longrightarrow r\sin(\theta) = -6$ or $r = -6\csc(\theta)$

$x - 2y = 3 \longrightarrow r\cos(\theta) - 2r\sin(\theta) = 3$

$x^2 + (y-1)^2 = 3 \rightarrow r^2\cos^2(\theta) + (r\sin(\theta) - 1)^2 = 3$

$$\underbrace{r^2\cos^2(\theta) + r^2\sin^2(\theta)}_{r^2} - 2r\sin(\theta) + 1 = 3$$

$$r^2 - 2r\sin(\theta) = 2$$

Sketch the polar curve.

3 petal flower

4 petal flower

$r = \cos 3\theta$

$r = \sin 2\theta$

You

dent

up/down

$r = 2 + \sin\theta$

$r = -1 + 2\cos\theta$

inner loop

left/right

$r = 1 + 2\sin\theta$

$r = 4$

Circle

centered at $(0,0)$

radius $= 4$

$\theta = -\frac{1}{4}\pi$

inner loop

up/down

line

$y = -x$

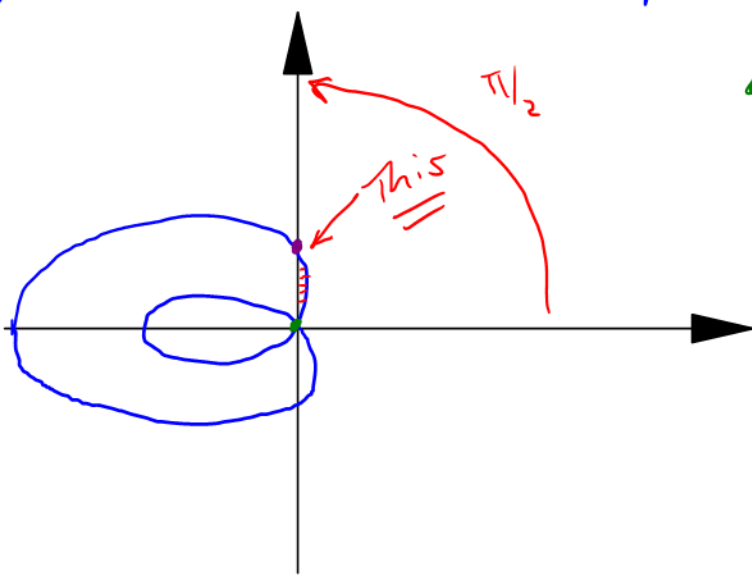
Give the area inside one petal of $r = 3\cos(4\theta)$.

you (done in class notes)

Give the area in the 1st quadrant that is inside the polar curve $r = 3 - 6\cos(\theta)$.

(*)

inner loop.



• = origin

$\Rightarrow r = 0$

Solve

$3 - 6\cos(\theta) = 0$

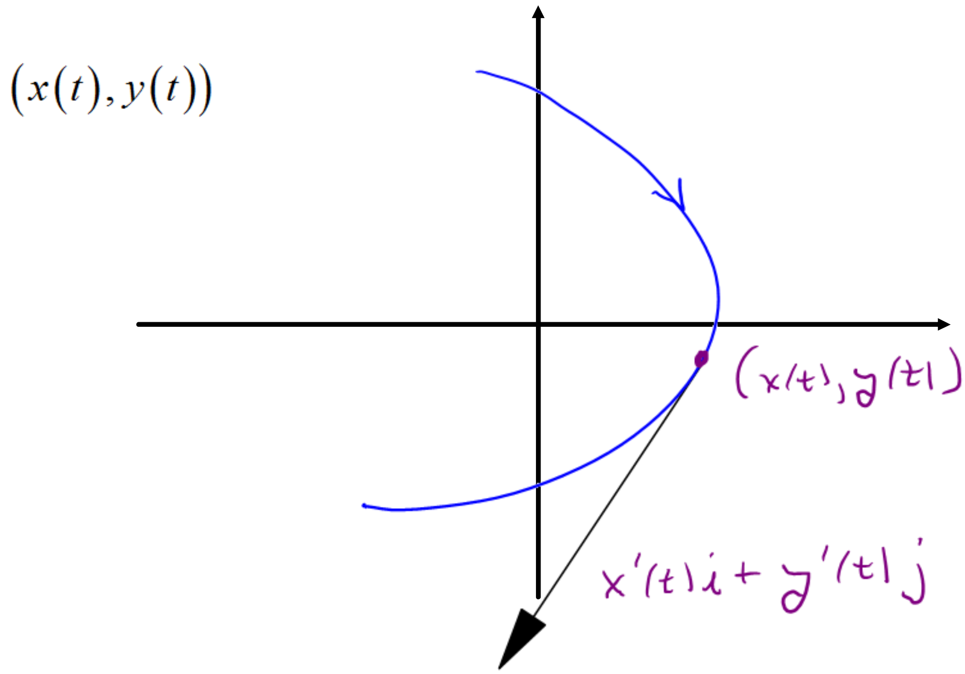
$\cos(\theta) = \frac{1}{2}$

$\theta = \pi/3$

• = $\boxed{\theta = \pi/2}$

$$\text{Area}(\;) = \int_{\pi/3}^{\pi/2} \frac{1}{2}\left(3 - 6\cos(\theta)\right)^2 d\theta$$

Part V - Parametric Equations



Express the parametric curve by an equation in x and y.

$x(t) = 2t - 1, \quad y(t) = t^2 + 1$

$t = \frac{x+1}{2} \Rightarrow \boxed{y = \left(\frac{x+1}{2}\right)^2 + 1}$

$x(t) = 3\sin(t), \quad y(t) = \cos(t)$

$\frac{x}{3} = \sin(t), \quad y = \cos(t)$

$\sin^2(t) + \cos^2(t) = 1$

$\left(\frac{x}{3}\right)^2 + y^2 = 1$

$\frac{x^2}{9} + y^2 = 1.$

Find a parametrization

$$x = x(t), \quad y = y(t), \quad t \in [0, 1].$$

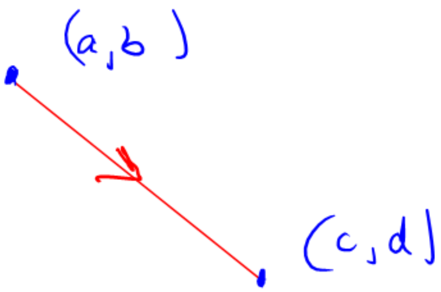
for the given curve.

$$x = 3 + t(5), \quad y = 7 + t(-2), \quad 0 \le t \le 1$$

The line segment from $(3, 7)$ to $(8, 5)$.

The line segment from $(2, 6)$ to $(6, 3)$.

The parabolic arc $x = 1 - y^2$ from $(0, -1)$, to $(0, 1)$. You



From (a,b) to (c,d)

$$x = a + t(c - a), \quad y = b + t(d - b), \quad 0 \le t \le 1.$$

Give tangent line and normal line to the curve at the point associated with the given value of t, using both xy equations and parametric equations.

$$x(t)=2-3\sin(t),\ y(t)=\cos(3t),\ t=\frac{\pi}{4}$$

Tangent Lines

xy Eq'ns

Point $= \left(x(\pi/4), y(\pi/4)\right) = \left(2-\frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Slope $= \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{-3\frac{\sqrt{2}}{2}}{-3\frac{\sqrt{2}}{2}} = 1$

$x'(t) = -3\cos(t)$

$y'(t) = -3\sin(3t)$

Equation: $y + \frac{\sqrt{2}}{2} = \left(x - \left(2-\frac{3\sqrt{2}}{2}\right)\right)$

Parametric Form:

$$x = x(\pi/4) + t\,x'(\pi/4)$$

$$y = y(\pi/4) + t\,y'(\pi/4)$$

i.e.

$$x = 2 - \frac{3\sqrt{2}}{2} + t\left(-\frac{3\sqrt{2}}{2}\right)$$

$$y = \frac{-\sqrt{2}}{2} + t\left(-3\frac{\sqrt{2}}{2}\right)$$

You can do normal eqns.

Find the points (x, y) at which the curve has: (a) a horizontal tangent; (b) a vertical tangent. Then sketch the curve.

$$x(t) = 3t - t^3, \quad y(t) = t + 1.$$
$$x(t) = t^2 - 2t, \quad y(t) = t^3 - 12t.$$
$$x(t) = 3 - 4\sin t, \quad y(t) = 4 + 3\cos t.$$
$$x(t) = \sin 2t, \quad y(t) = \sin t.$$
$$x(t) = t^2 - 2t, \quad y(t) = t^3 - 3t^2 + 2t.$$
$$x(t) = 2 - 5\cos t, \quad y(t) = 3 + \sin t.$$

Horiz tangent: Need slope $= 0$.

$\frac{y'}{x'} = 0$ i.e. $y' = 0$ and $x' \neq 0$.

Note: $y' = 3t^2 - 6t + 2$ $\quad x' = 2t - 2$

Solve $y' = 0$. $\quad 3t^2 - 6t + 2 = 0$

Quad formula $\quad t = \frac{6 \pm \sqrt{36 - 24}}{6}$

$t = 1 \pm \frac{\sqrt{12}}{6}$

Vert tangents: $\frac{x'}{y'} = 0$ i.e. $x' = 0$ and $y' \neq 0$.

Solve $2t - 2 = 0 \Rightarrow t = 1$

Give a formula for the length of the curve given by

$x(t)=2-3\sin(t),\ y(t)=\cos(3t)$

No interval?

2π periodic.

Parametric ||

$$\int_0^{2\pi}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

$y=x^2-2x,\ \ 1\le x\le 3$

$y=f(x)$ form |||

$$\int_1^3\sqrt{1+(f'(x))^2}\,dx$$

$r=2+3\cos(\theta),\ \frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$

|| polar form

$$\int_{\pi/4}^{3\pi/4}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

The equations below give the position of a particle at each time t during the time interval specified. Find the initial speed of the particle, the terminal speed of the partical, and the distance traveled by the particle.

$$x(t) = t^2, \quad y(t) = 2t, \quad \text{from } t = 0 \text{ to } t = \sqrt{3}.$$

$$x(t) = t - 1, \quad y(t) = \tfrac{1}{2}t^2, \quad \text{from } t = 0 \text{ to } t = 1.$$

$$x(t) = t^2, \quad y(t) = t^3, \quad \text{from } t = 0 \text{ to } t = 1.$$

initial speed = magnitude of initial velocity.

at $t=0$

terminal speed = magnitude of terminal velocity

at $t=1$

Note:

$$|ai+bj| = \sqrt{a^2+b^2}$$

$$\text{velocity} = v(t) = x'(t)\,i + y'(t)\,j$$

$$= 2t\,i + 3t^2\,j$$

$$\text{initial speed} = |v(0)| = |0i + 0j| = 0$$

$$\text{terminal speed} = |v(1)| = |2i + 3j| = \sqrt{4+9}$$

$$= \sqrt{13}$$

$$\text{Distance Traveled} = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^1 \sqrt{4t^2 + 9t^4}\, dt = \int_0^1 t\sqrt{4+9t^2}\, dt$$

... u-subst.

Part VI - Sets, Sequences, LUB, GLB, Monotonicity and Limits

Find the $\overbrace{\text{least upper}}^{\text{LUB}}$ bound (if it exists) and the $\overbrace{\text{greatest lower}}^{\text{GLB}}$ bound of the given set.

$\{x : x^2 < 4\}$.

$\{x : x^3 \geq 8\}$.

$\{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots\}$.

$\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\}$.

$\{x : |x - 1| < 2\}$.

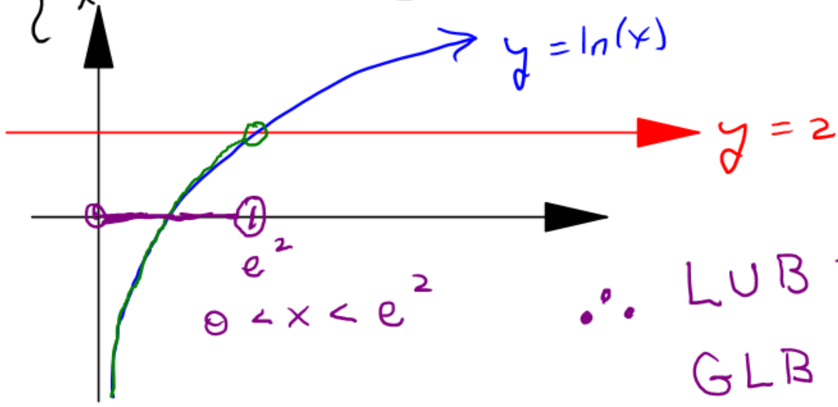
$\{x : x^4 \leq 16\}$.

$\{x : x^2 + x + 2 \geq 0\}$.

$\{x : \ln|x| < 2\} = (0, e^2)$

$\ln(x) = 2$

$x = e^2$



$0 < x < e^2$

$\therefore$ LUB $= e^2$

GLB $= 0$

Note: limit $\Rightarrow$ bounded.

Determine the boundedness and monotonicity of the sequence with a_n as indicated.

monotone? $f(x) = \frac{2}{x}$, $x \geq 1$

$f'(x) = \frac{-2}{x^2} < 0$

$\Rightarrow f$ is decreasing $\Rightarrow$ sequence is decreasing. $\therefore$ monotone!

$\frac{2}{n}$. $\rightarrow 0$ $\therefore$ bounded

$\frac{n+(-1)^n}{n} = 1 + \frac{(-1)^n}{n} \rightarrow 1$ $\therefore$ bounded.

$0, \frac{3}{2}, \frac{2}{3}$ Not monotone

$(0.9)^n$. $\rightarrow 0$ $\therefore$ bounded. decreases. $\therefore$ monotone.

$\sqrt{n^2+1}$. Not bounded increasing!

$f(x) = \sqrt{x^2+1}$, $x \geq 1$

$f'(x) = \frac{x}{\sqrt{x^2+1}} > 0$

$\Rightarrow f$ is increasing $\Rightarrow$ sequence increasing $\Rightarrow$ monotone.

$\frac{2^n}{4^n+1}$.

$\frac{n^2}{\sqrt{n^3+1}}$.

$\frac{n+2}{3^{10}\sqrt{n}}$.

$(-1)^n\sqrt{n}$.

$\ln\left(\frac{n+1}{n}\right)$.

$\frac{\sqrt{n+1}}{\sqrt{n}}$.

$\frac{(-1)^n}{n}$.

$(1.001)^n$.

$\frac{n-1}{n}$.

$\frac{n^2}{n+1}$.

$\frac{4n}{\sqrt{4n^2+1}}$.

$\frac{4^n}{2^n+100}$.

$\ln\left(\frac{2n}{n+1}\right)$.

See the 2011 review video. All there.