Test 3 Review Problems

Sections 8.4 - 10.3

10 questions: 3 multiple choice and 7 written

- Trigonometric Substitution
- Partitial Fraction Decomposition
- Numerical Integration
- Polar Coordinates
- Parametric Curves
- Sets, Sequences, LUB, GLB, Monotonicity and Limits

Spring 2011

Part I - Trigonometric Substitution

$a > 0$

$\sqrt{a^2 - x^2} \longleftrightarrow x = a\sin(\theta)$

$\sqrt{a^2 + x^2} \longleftrightarrow x = a\tan(\theta)$

$\sqrt{x^2 - a^2} \longleftrightarrow x = a\sec(\theta)$

Process: * 1. Identify + make the substitution.

2. Integrate

* 3. Convert back to x.

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3(\theta)\cdot 2\sec^2(\theta)\,d\theta}{\sqrt{4+4\tan^2(\theta)}}$$

$x = 2\tan(\theta)$

$dx = 2\sec^2(\theta)\,d\theta$

$$= \int \frac{16\tan^3(\theta)\sec^2(\theta)}{2\sqrt{1+\tan^2(\theta)}}\,d\theta$$

1

$$= \int 8\,\frac{\tan^3(\theta)\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= \int 8\tan^3(\theta)\sec(\theta)\,d\theta$$

2

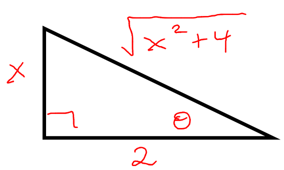
$$= 8\int \tan^2(\theta)\,\underline{\sec(\theta)\tan(\theta)\,d\theta}$$

$$= 8\int \left(\sec^2(\theta)-1\right)\sec(\theta)\tan(\theta)\,d\theta$$

$$= 8\int \sec^2(\theta)\,\underline{\sec(\theta)\tan(\theta)\,d\theta} - 8\int \sec(\theta)\tan(\theta)\,d\theta$$

$$= 8\cdot\frac{1}{3}\sec^3(\theta) - 8\sec(\theta) + C$$

$x = 2\tan(\theta) \Leftrightarrow \frac{x}{2} = \tan(\theta)$



$\Rightarrow \sec(\theta) = \frac{\sqrt{x^2+4}}{2}$

$$= \frac{8}{3}\cdot\frac{(x^2+4)^{3/2}}{8} - 8\cdot\frac{\sqrt{x^2+4}}{2} + C$$

$$= \frac{1}{3}\cdot(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

3

$$\int x^2\sqrt{25-x^2}\,dx = \int 25\sin^2(\theta)\sqrt{25-25\sin^2(\theta)}\,5\cos(\theta)d\theta$$

$x = 5\sin(\theta)$

$dx = 5\cos(\theta)d\theta$

$$= 625\int \sin^2(\theta)\sqrt{\cos^2(\theta)}\cos(\theta)d\theta$$

$$= 625\int \sin^2(\theta)\cos^2(\theta)\,d\theta$$

1

$$= 625\int \left(\sin(\theta)\cos(\theta)\right)^2 d\theta$$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\frac{1}{2}\sin(2\theta) = \sin(\theta)\cos(\theta)$

$$= 625\int \frac{1}{4}\sin^2(2\theta)\,d\theta$$

$$= \frac{625}{4}\int\left[\frac{1}{2} - \frac{\cos(4\theta)}{2}\right]d\theta$$

$$= \frac{625}{8}\theta - \frac{625}{8}\cdot\frac{1}{4}\sin(4\theta) + C$$

2

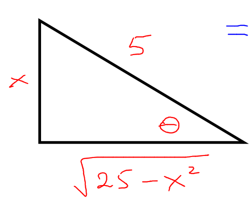
$x = 5\sin(\theta) \leftrightarrow \boxed{\frac{x}{5} = \sin(\theta)}$

$\arcsin\left(\frac{x}{5}\right) = \theta \leftarrow$

$\sin(4\theta) = \sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta)$

$= 2\cdot 2\sin(\theta)\cos(\theta)\left(\cos^2(\theta) - \sin^2(\theta)\right)$

$= 4\sin(\theta)\cos(\theta)\left(\cos^2(\theta) - \sin^2(\theta)\right)$



$\rightarrow \cos(\theta) = \frac{\sqrt{25-x^2}}{5}$

$\sin(\theta) = \frac{x}{5}$

Substituting these here

and $\theta = \arcsin\left(\frac{x}{5}\right)$

earlier leads to

the answer.

Part II - Partial Fraction Decomposition

Setting: $\displaystyle\int \frac{p(x)}{q(x)}\,dx$

degree$(p(x))$ < degree$(q(x))$

1. Use partial fraction decomp to rewrite

$$\frac{p(x)}{q(x)} = \underbrace{\cdots\cdots}$$

2. Integrate

$$\int \frac{p(x)}{q(x)}\,dx = \int \boxed{}\,dx$$

$$\int \frac{7}{(x-2)(x+5)}\,dx.$$

1. Do partial fraction decomp.

on $\frac{7}{(x-2)(x+5)}$

$$\frac{7}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$7 = A(x+5) + B(x-2)$$

$\underline{\underline{x=2}}$ $\quad 7 = 7A \quad \boxed{A=1}$

$\underline{x=-5}$ $\quad 7 = -7B \quad \boxed{B=-1}$

$\therefore$ The pfd is:

$$\frac{7}{(x-2)(x+5)} = \frac{1}{x-2} + \frac{-1}{x+5}$$

$\Rightarrow$

$$\int \frac{7}{(x-2)(x+5)}dx = \int\left[\frac{1}{x-2} + \frac{-1}{x+5}\right]dx$$

$$= \ln|x-2| - \ln|x+5| + C$$

$$= \ln\left|\frac{x-2}{x+5}\right| + C$$

#

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx$$

$\underbrace{x^2+1}_{\text{irred. quad}}$

1. Do pfd on $\dfrac{x^2+5x+2}{(x+1)(x^2+1)}$

$$\frac{x^2+5x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$x^2+5x+2 = A(x^2+1) + (Bx+C)(x+1)$$

$\underline{\underline{x=-1}}$ $\quad -2 = 2A \qquad \boxed{A=-1}$

Grocery shop

$$\underline{\underline{x^2}}+5x+2 = -x^2-1 + Bx^2 + (B+C)x + C$$

$\underline{\underline{x^2 \text{ terms}}}$: $\quad 1 = -1 + B \Rightarrow \boxed{B=2}$

$\underline{x \text{ terms}}$: $\quad 5 = B + C$

$\quad 5 = 2 + C \Rightarrow \boxed{C=3}$

$\therefore$ the pfd is

$$\frac{x^2+5x+2}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{2x+3}{x^2+1}$$

$\underline{\underline{\text{Integrate}}}$:

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left[\frac{-1}{x+1} + \frac{2x+3}{x^2+1}\right]dx$$

$$= -\ln|x+1| + \int \frac{2x+3}{x^2+1}\,dx$$

$$= -\ln|x+1| + \int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx$$

$$= -\ln|x+1| + \ln|x^2+1| + 3\arctan(x) + C$$

$$\int \frac{1}{x(x-1)^2}\,dx$$

1. Get the pfd of

$$\frac{1}{x(x-1)^2}.$$

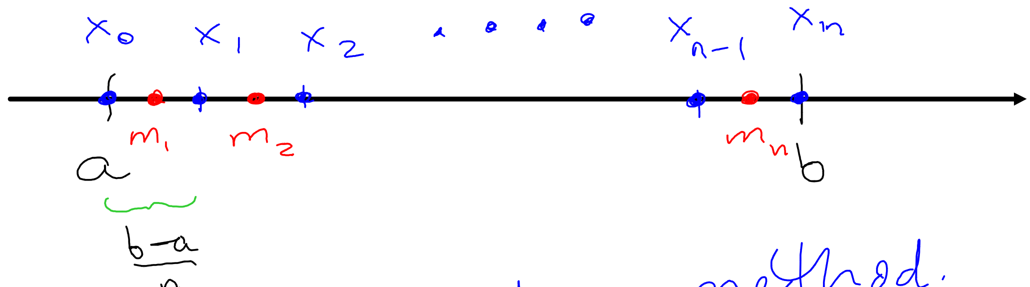
$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$\vdots$

Part III - Numerical Integration

Tool for approximating

$$\int_a^b f(x)\,dx$$



1. Choose n and a method.

2. $T_n = \left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)\frac{b-a}{2n}$

$$M_n = \left(f(m_1) + f(m_2) + \cdots + f(m_n)\right)\frac{b-a}{n}$$

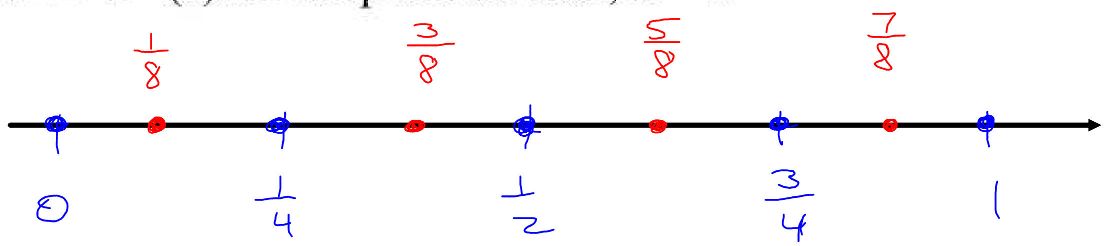
$$S_n = \frac{1}{3}T_n + \frac{2}{3}M_n$$

Estimate

$$\int_0^1 \frac{dx}{1+x^2} \qquad f(x) = \frac{1}{1+x^2}$$

by: (a) the trapezoidal rule, $\boxed{n = 4}$; (b) Simpson's rule, $n = 4$. (c) the midpoint estimate, $n = 4$.

1 2 3



$$T_4 = \left(f(0) + 2f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 2f\left(\tfrac{3}{4}\right) + f(1)\right)\frac{1-0}{2\cdot 4}$$

$$= \left(1 + 2\cdot\frac{16}{17} + 2\cdot\frac{4}{5} + 2\cdot\frac{16}{25} + \frac{1}{2}\right)\cdot\frac{1}{8}$$

$$f(x) = \frac{1}{1+x^2}$$

$$M_4 = \left(f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right)\right)\frac{1-0}{4}$$

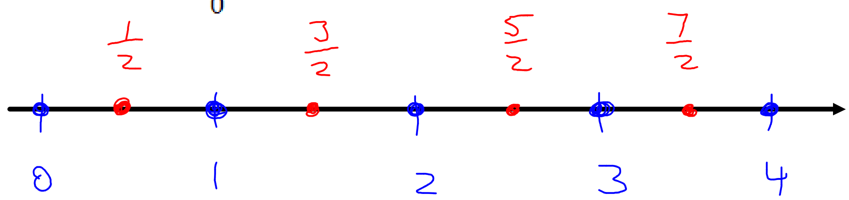
$$= \left(\frac{64}{65} + \frac{64}{73} + \frac{64}{89} + \frac{64}{113}\right)\cdot\frac{1}{4}$$

$$S_4 = \frac{1}{3}T_4 + \frac{2}{3}M_4 = \underline{\underline{\text{answer}}}$$

Use the midpoint, trapezoid and Simpson's methods to

estimate $\int_0^4 f(x)\,dx$ with $n = 4$.

x	$f(x)$
0	1
.25	.8
.5	.7
.75	.7
1	.6
1.25	.5
1.5	.4
1.75	.3
2	.3
2.25	.2
2.5	.1
2.75	0
3	0
3.25	.1
3.5	.2
3.75	.3
4	.4



$$M_4 = \left(f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + f(\tfrac{7}{2})\right)\frac{4-0}{4}$$

$$= (.7 + .4 + .1 + .2)\cdot 1$$

$$= 1.4$$

$$T_4 = \left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right)\frac{4-0}{2\cdot 4}$$

$$= \left(1 + 2\cdot .6 + 2\cdot .3 + 2\cdot 0 + .4\right)\cdot\frac{1}{2}$$

$$= (3.2)\cdot\frac{1}{2} = 1.6$$

$$S_4 = \frac{1}{3}T_4 + \frac{2}{3}M_4 \quad = \underline{\underline{\text{answer}}}$$

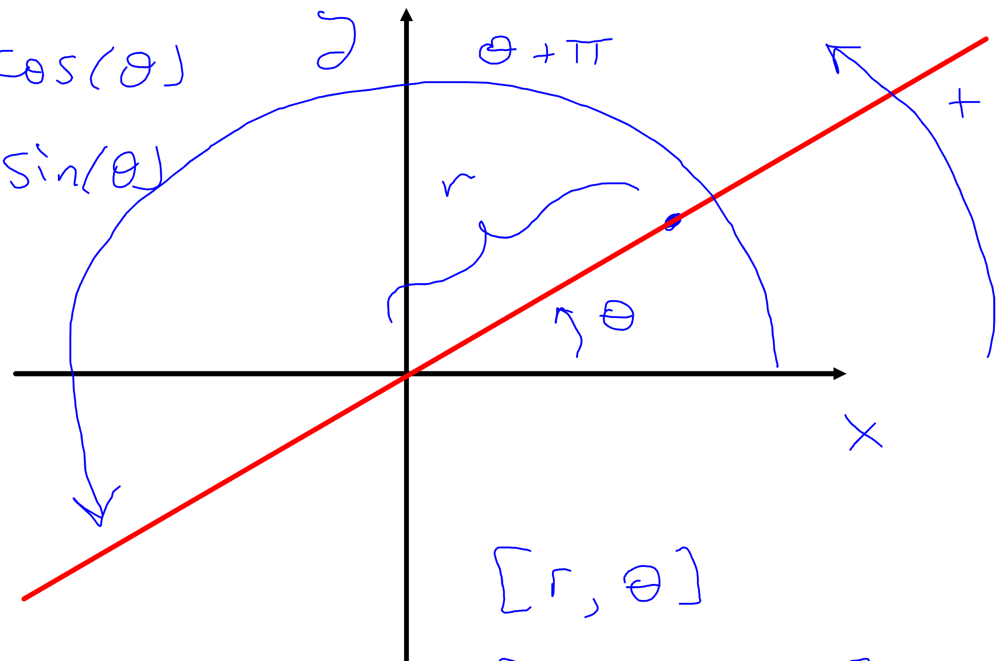
$$= \frac{1}{3}\cdot(1.6) + \frac{2}{3}(1.4)$$

$$= \frac{1}{3}\left[1.6 + 2.8\right] = \frac{4.4}{3}. \quad \#$$

Part IV - Polar Coordinates

$x = r\cos(\theta)$

$y = r\sin(\theta)$



$[r, \theta]$

Many more $\quad [-r, \theta + \pi]$

$[r, \theta + 2\pi]$

Standard rep: $\quad r = \sqrt{x^2 + y^2}$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

if (x, y) in Q1 or Q4.

Find the rectangular coordinates of the point.

$[3, \frac{1}{2}\pi]$. $[4, \frac{1}{6}\pi]$.
$[-1, -\pi]$. $[-1, \frac{1}{4}\pi]$.

Polar $[r, \theta]$

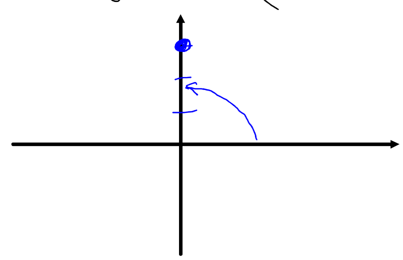
Easy: (x, y)

where $x = r\cos(\theta)$
$y = r\sin(\theta)$

$[3, \frac{\pi}{2}]$ $x = 3\cos(\pi/2) = 0$

$y = 3\sin(\pi/2) = 3$

$(x, y) = (0, 3)$.

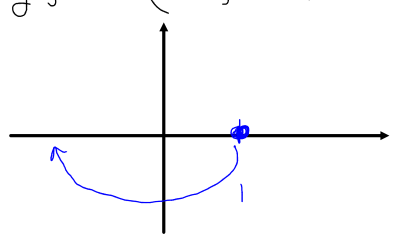


$[-1, -\pi]$ $x = r\cos(\theta)$
$y = r\sin(\theta)$

$x = -1 \cdot \cos(-\pi) = 1$

$y = -1 \cdot \sin(-\pi) = 0$

$(x, y) = (1, 0)$



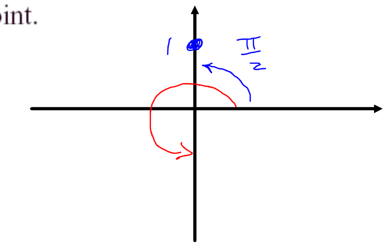
Points are specified in rectangular coordinates. Give all possible polar coordinates for each point.

✓ $(0,1)$. $\parallel \longrightarrow$

✓ $(-3,0)$.

$(2,-2)$.

✓ $(4\sqrt{3},4)$.



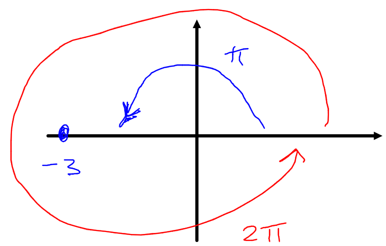
$[1, \frac{\pi}{2}] \longrightarrow [1, \frac{\pi}{2} + 2\pi k]$

$k = 0, \pm 1, \pm 2, \ldots$

$[-1, \frac{3\pi}{2}] \to [-1, \frac{3\pi}{2} + 2\pi k]$

$k = 0, \pm 1, \pm 2, \ldots$

$(-3, 0)$



$[3, \pi] \longrightarrow [3, \pi + 2\pi k]$

$[-3, 2\pi] \to [-3, 2\pi + 2\pi k]$

$k = 0, \pm 1, \pm 2, \ldots$

$(4\sqrt{3}, 4)$ Is in Q1.

$$r = \sqrt{(4\sqrt{3})^2 + 4^2} = 4\sqrt{3+1} = 8$$

$$\theta = \arctan\left(\frac{4}{4\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= \pi/6$$

$[8, \frac{\pi}{6}] \longrightarrow [8, \frac{\pi}{6} + 2\pi k]$

$[-8, \frac{7\pi}{6}] \to [-8, \frac{7\pi}{6} + 2\pi k]$

$k = 0, \pm 1, \pm 2, \ldots$

Write the equation in polar coordinates.

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x = 2.$

$2xy = 1.$

$x^2 + (y-2)^2 = 4.$

$y = 3.$

$x^2 + y^2 = 9.$

$r\cos(\theta) = 2 \Rightarrow r = 2\sec(\theta)$

$r\sin(\theta) = 3 \Rightarrow r = 3\csc(\theta)$

$2r\cos(\theta)r\sin(\theta) = 1$

$r^2\sin(2\theta) = 1$

$r = 3$

Sketch the polar curve.

$r = \cos 3\theta$ 3 petal flower

$r = 2 + \sin\theta$ dent

$r = 1 + 2\sin\theta$

$\theta = -\frac{1}{4}\pi$

$r = \sin 2\theta$ 4 petal flower

$r = -1 + 2\cos\theta$ inner loop.

$r = 4$

$r = a + b\sin(\theta)$

$r = a + b\cos(\theta)$

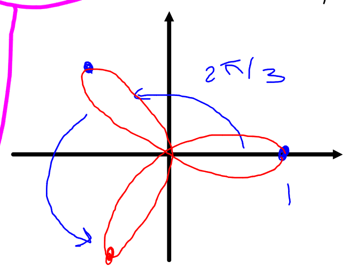
$r = a\cos(\theta)$ $\underline{a>0}$

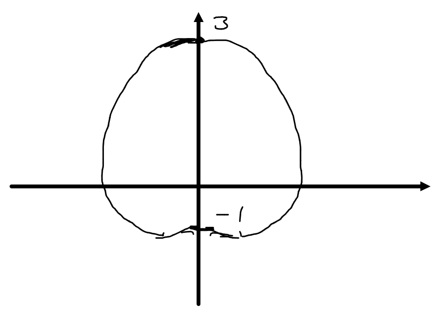
$r = a\sin(\theta)$

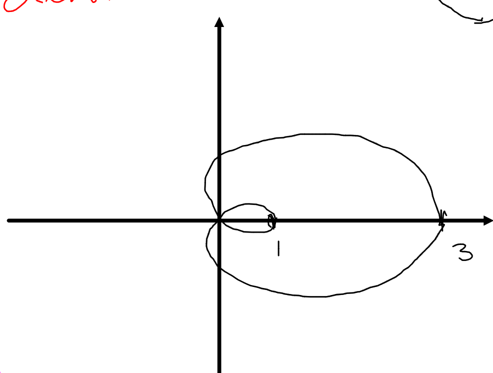
$r = r_0$

$r = a\cos(m\theta)$

$r = a\sin(m\theta)$

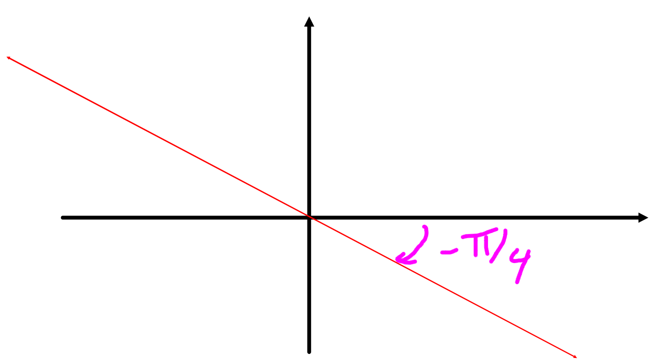






$r = -1 + 2\cos(\theta)$

$\theta = -\pi/4$



Give the area:

Inside one petal of $r = 2\sin 3\theta$

Inside the inner loop of $r = 1 - 2\sin\theta$

Outside $r = 1 + \cos\theta$, but inside $r = 2 - \cos\theta$

Inside $r = 2$, but outside $r = 4\cos\theta$

$1 + \cos(\theta) = 2 - \cos(\theta)$

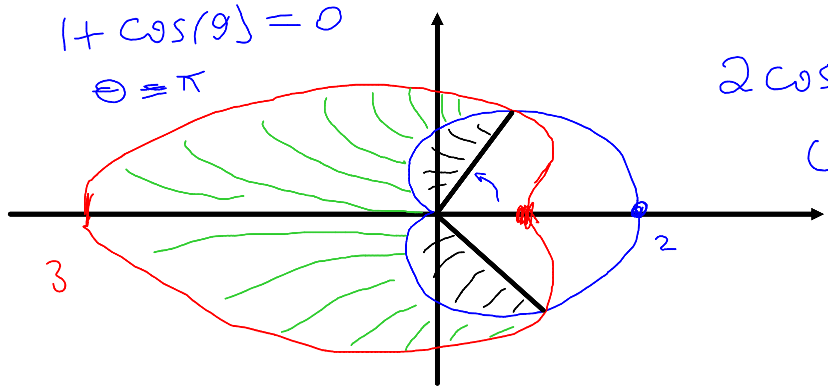
$2\cos(\theta) = 1$

$\cos(\theta) = \frac{1}{2}$

$\theta = \pi/3$

$1 + \cos(\theta) = 0$

$\theta = \pi$



involves $r = 2 - \cos(\theta)$

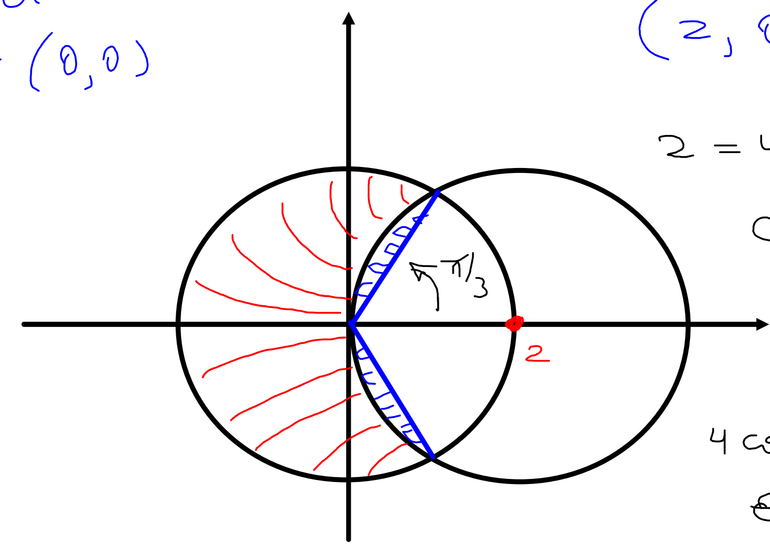
involves $r = 1 + \cos(\theta)$

$$= 2\int_{\pi/3}^{\pi} \frac{1}{2}\left(2 - \cos(\theta)\right)^2 d\theta - 2\int_{\pi/3}^{\pi} \frac{1}{2}\left(1 + \cos(\theta)\right)^2 d\theta$$

Inside $r = 2$, but outside $r = 4\cos\theta$

circle
rad. 2
centered
at $(0,0)$

circle
radius 2
centered at
$(2,0)$



$2 = 4\cos(\theta)$

$\cos(\theta) = \frac{1}{2}$

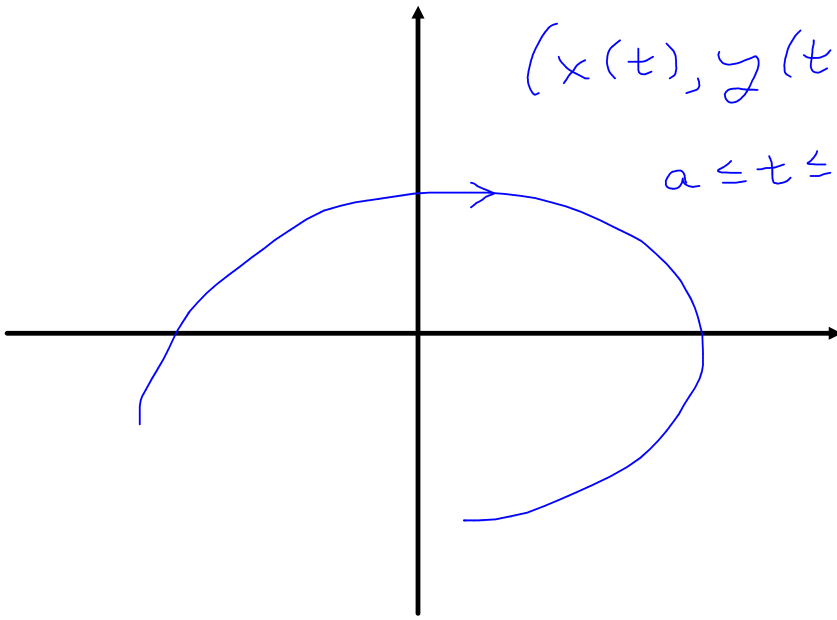
$\theta = \pi/3$

$4\cos(\theta) = 0$

$\theta = \pi/2$

$$= 2\int_{\pi/3}^{\pi} \frac{1}{2}\left(2\right)^2 d\theta - 2\int_{\pi/3}^{\pi/2} \frac{1}{2}\left(4\cos(\theta)\right)^2 d\theta$$

Part V - Parametric Equations



$(x(t), y(t))$

$a \le t \le b$

Express the curve by an equation in x and y.

$x(t) = t^2, \quad y(t) = 2t + 1.$

$x(t) = 3t - 1, \quad y(t) = 5 - 2t.$

$x(t) = t^2, \quad y(t) = 4t^4 + 1.$

$x(t) = 2t - 1, \quad y(t) = 8t^3 - 5.$

$x(t) = 2\cos t, \quad y(t) = 3\sin t.$

$x(t) = \sec^2 t, \quad y(t) = 2 + \tan t.$

$t = \frac{1}{3}(x+1)$

$y = 5 - 2 \cdot \frac{1}{3}(x+1)$

$y = 5 - \frac{2}{3}x - \frac{2}{3}$

$\boxed{y = \frac{13}{3} - \frac{2}{3}x}$

$t = \frac{1}{2}(y-1)$

$\Rightarrow x = t^2 = \left(\frac{1}{2}(y-1)\right)^2$

$\boxed{x = \frac{1}{4}(y-1)^2}$

$x = t^2, \quad y = 4t^4 + 1$

$= 4(t^2)^2 + 1$

$y = 4x^2 + 1, \quad x \geq 0.$

$x(t) = 2\cos t, \quad y(t) = 3\sin t.$

$\frac{x}{2} = \cos(t) \qquad \frac{y}{3} = \sin(t)$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

Find a parametrization

$$x = x(t), \quad y = y(t), \quad t \in [0, 1].$$

for the given curve.

- The line segment from $(3, 7)$ to $(8, 5)$.
- The line segment from $(2, 6)$ to $(6, 3)$.
- The parabolic arc $x = 1 - y^2$ from $(0, -1)$, to $(0, 1)$.

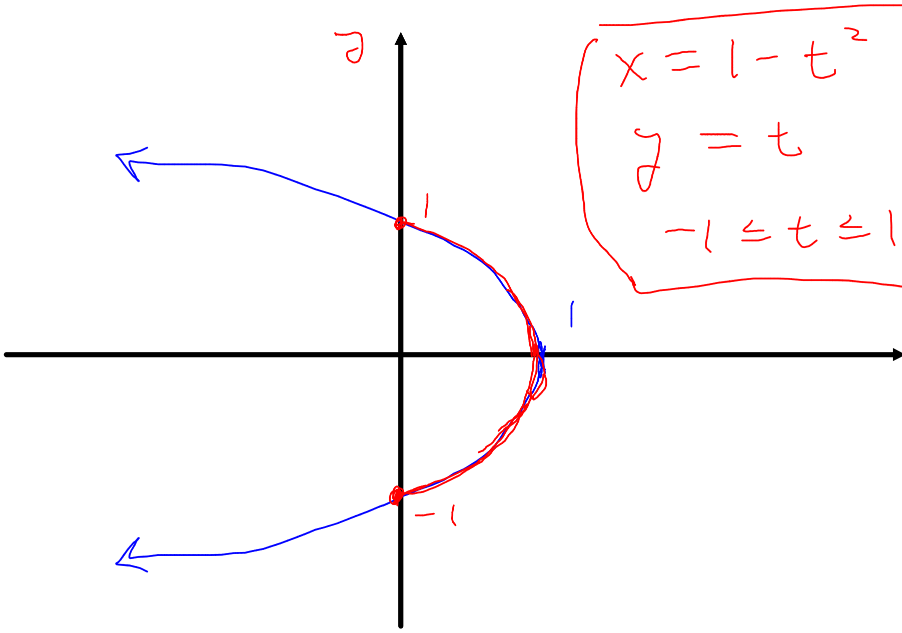
From (a, b) to (c, d)

$$\boxed{\begin{aligned} x &= a + t(c - a) \\ y &= b + t(d - b) \end{aligned} \quad 0 \le t \le 1}$$

$$\begin{aligned} x &= 3 + t(8 - 3) \\ y &= 7 + t(5 - 7) \end{aligned} \quad 0 \le t \le 1$$

$$\boxed{\begin{aligned} x &= 3 + 5t \\ y &= 7 - 2t \end{aligned} \quad , \quad 0 \le t \le 1}$$

The parabolic arc $x = 1 - y^2$ from $(0, -1)$, to $(0, 1)$.



$x = 1 - t^2$
$y = t$
$-1 \le t \le 1.$

Give equations for the tangent line and normal line to the curve at the point associated with the given value of t.

$x(t) = t, \quad y(t) = t^3 - 1; \quad t = 1.$ $\longrightarrow x' = 1, \; y' = 3t^2$

$x(t) = t^2, \quad y(t) = t + 5; \quad t = 2.$

$x(t) = 2t, \quad y(t) = \cos \pi t; \quad t = 0.$

$x(t) = 2t - 1, \quad y(t) = t^4; \quad t = 1.$

$x(t) = t^2, \quad y(t) = (2 - t)^2; \quad t = \frac{1}{2}.$

$x(t) = 1/t, \quad y(t) = t^2 + 1; \quad t = 1.$

$x(t) = \cos^3 t, \quad y(t) = \sin^3 t \quad t = \frac{1}{4}\pi.$

$x(t) = e^t, \quad y(t) = 3e^{-t}; \quad t = 0.$

Point: $(x(1), y(1)) = (1, 0)$

T.L. Slope: $\dfrac{y'(1)}{x'(1)} = \dfrac{3}{1} = 3$

N.L. Slope: $-\dfrac{1}{3}$

T.L. Eq: $y - 0 = 3(x - 1)$

N.L. Eq: $y - 0 = -\frac{1}{3}(x - 1)$

Find the points (x, y) at which the curve has: (a) a horizontal tangent; (b) a vertical tangent. Then sketch the curve.

$$x(t) = 3t - t^3, \quad y(t) = t + 1.$$
$$x(t) = t^2 - 2t, \quad y(t) = t^3 - 12t.$$
$$x(t) = 3 - 4\sin t, \quad y(t) = 4 + 3\cos t.$$
$$x(t) = \sin 2t, \quad y(t) = \sin t.$$
$$x(t) = t^2 - 2t, \quad y(t) = t^3 - 3t^2 + 2t.$$
$$x(t) = 2 - 5\cos t, \quad y(t) = 3 + \sin t.$$

Horiz. Tangent: $\frac{y'}{x'} = 0$

Vert Tangent: $\frac{x'}{y'} = 0$

$x' = 2\cos(2t) \quad y' = \cos(t)$

H.T. Set $\frac{y'}{x'} = 0 \quad \frac{\cos(t)}{2\cos(2t)} = 0$

$t = \pi/2, \quad t = -\frac{\pi}{2}$

$t = \pi/2$ Point: $\left(\sin\left(2 \cdot \frac{\pi}{2}\right), \sin(\pi/2)\right)$

$(0, 1)$

$t = -\frac{\pi}{2}$ Point: $\left(\sin\left(2 \cdot \left(-\frac{\pi}{2}\right)\right), \sin(-\pi/2)\right)$

$= (0, -1)$

V.T. Solve $\frac{x'}{y'} = 0$

$\frac{2\cos(2t)}{\cos(t)} = 0$

$t = \pi/4, \ 3\pi/4, \ -\pi/4, \ -3\pi/4$

Points: Do it.

Find the length of the graph and compare it to the straight-line distance between the endpoints of the graph.

$$f(x) = (x - \tfrac{4}{9})^{3/2}, \quad x \in [1, 4].$$

$$f(x) = x^{3/2}, \quad x \in [0, 44].$$

$$f(x) = \tfrac{1}{3}\sqrt{x}\,(x - 3), \quad x \in [0, 3].$$

$y = f(x)$, $a \leq x \leq b$

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$(x(t), y(t))$, $a \leq t \leq b$

$$\text{length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$f(x) = \tfrac{1}{3}\sqrt{x}\,(x-3), \quad x \in [0,3].$$

$$f(x) = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{length} = \int_0^3 \sqrt{1 + \left(\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\right)^2}\,dx$$

$$= \int_0^3 \sqrt{1 + \frac{1}{4}x - 2\cdot\frac{1}{4} + \frac{1}{4}x^{-1}}\,dx$$

$$= \int_0^3 \sqrt{\frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1}}\,dx$$

$$= \int_0^3 \sqrt{\left(\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)^2}\,dx$$

$$= \int_0^3 \left(\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)dx.$$

The equations below give the position of a particle at each time t during the time interval specified. Find the initial speed of the particle, the terminal speed of the partical, and the distance traveled by the particle.

$$x(t) = t^2, \quad y(t) = 2t, \quad \text{from } t = 0 \text{ to } t = \sqrt{3}.$$
$$x(t) = t - 1, \quad y(t) = \tfrac{1}{2}t^2, \quad \text{from } t = 0 \text{ to } t = 1.$$
$$x(t) = t^2, \quad y(t) = t^3, \quad \text{from } t = 0 \text{ to } t = 1.$$

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$$
$$= \sqrt{(2t)^2 + (3t^2)^2}$$
$$= \sqrt{4t^2 + 9t^4}$$

$$\text{Initial Speed} = \sqrt{0 + 0} = 0$$
$$\text{Terminal Speed} = \sqrt{4 + 9} = \sqrt{13}$$

$$\text{Distance Traveled} = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$
$$= \int_0^1 \sqrt{4t^2 + 9t^4}\, dt$$
$$= \int_0^1 t\sqrt{4 + 9t^2}\, dt$$

Part VI - Sets, Sequences, LUB, GLB, Monotonicity and Limits

Find the least upper bound (if it exists) and the greatest lower
$\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\}$. the given set.

$\{x : x^2 < 4\}$.

$\{x : x^3 \geq 8\}$.

$\{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots\}$.

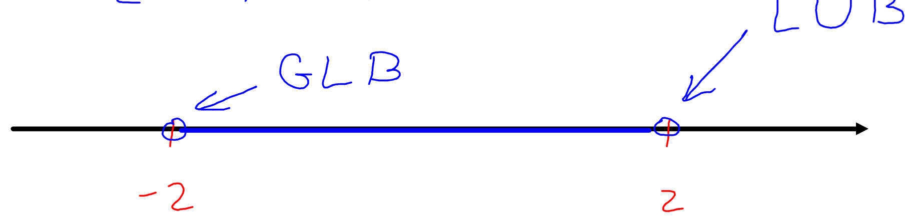
$\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\}$.

$\{x : |x-1| < 2\}$.

$\{x : x^4 \leq 16\}$.

$\{x : x^2 + x + 2 \geq 0\}$.

$-2 < x < 2$



$x^2 + x + 2 \geq 0$

$$x = \frac{-1 \pm \sqrt{1-8}}{2}$$

NO roots

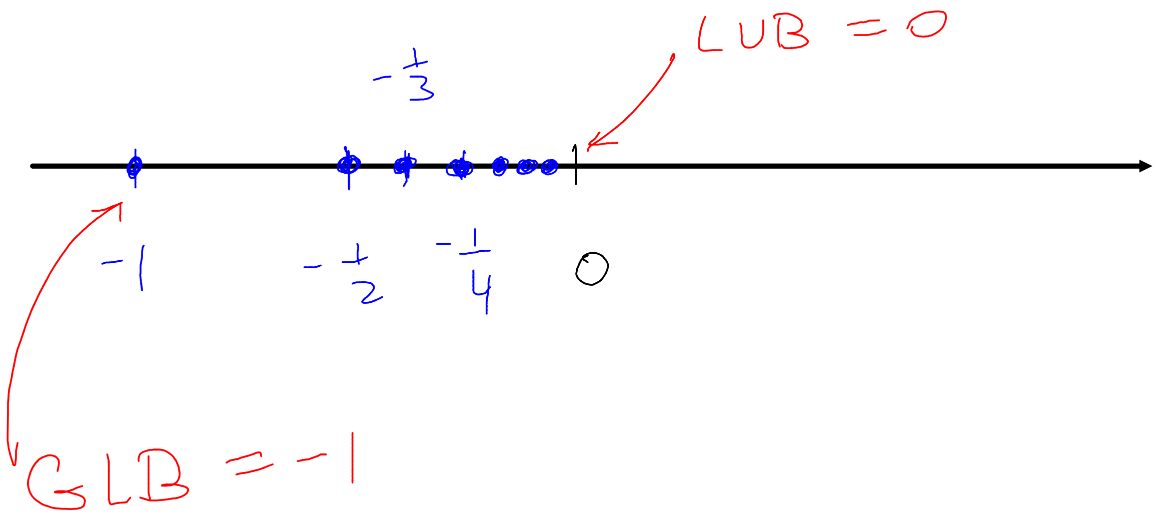
$\therefore \{x \mid x^2 + x + 2 \geq 0\}$

= all real #s.

LUB $\longleftrightarrow$ dne

GLB $\longleftrightarrow$ dne

$$\{-1, -\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{4}, \ldots\}.$$



Limit $\Rightarrow$ Bounded

Determine the boundedness and monotonicity of the sequence with a_n as indicated.

B $\equiv$ bounded

M $\equiv$ monotone

decreases $\therefore$ M

$\frac{2}{n}$. $\to 0$ $\therefore$ B

$\frac{(-1)^n}{n}$. $\to 0$ $\therefore$ B

$-,+,-,+,\ldots$ not M

$\frac{n+(-1)^n}{n}$. $= 1+\frac{(-1)^n}{n} \to 1$ $\therefore$ B

$0, \frac{3}{2}, \frac{2}{3}$ not M

$(1.001)^n$. $\to \infty$ $\therefore$ not Bounded

increases $\therefore$ M

$\frac{n-1}{n}$. $\to 1$ $\therefore$ B

Increasing $\therefore$ M

$f(x) = \frac{x-1}{x} = 1-\frac{1}{x}$, $x \geq 1$

$f'(x) = \frac{+1}{x^2}$

$(0.9)^n$. $\to 0$ $\therefore$ B

decreases $\therefore$ M

$\sqrt{n^2+1}$. $\sim \sqrt{n^2} = n \to \infty$ $\therefore$ not bounded

$\frac{n^2}{n+1}$. $\to \infty$ $\therefore$ not bounded

$\frac{2^n}{4^n+1}$. $\sim \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n \to 0$ $\therefore$ B

$\frac{4n}{\sqrt{4n^2+1}}$. $\sim \frac{4n}{2n} = 2$, $\to 2$ $\therefore$ B

$\frac{n^2}{\sqrt{n^3+1}}$. $\sim \frac{n^2}{n^{3/2}} \to \infty$ not bounded

$\frac{4^n}{2^n+100}$. $\to \infty$

$\frac{n+2}{3^{10}\sqrt{n}}$. $\sim \frac{n}{3^{10}\sqrt{n}} = \frac{\sqrt{n}}{3^{10}} \to \infty$ $\therefore$ not bdd

$\ln\left(\frac{2n}{n+1}\right)$. $\to \ln(2)$ $\therefore$ B

$(-1)^n\sqrt{n}$. osc. btwn $-\infty$ and ∞ $\Rightarrow$ not bdd

$\ln\left(\frac{n+1}{n}\right)$. $\to 0$ $\therefore$ B

$\frac{\sqrt{n+1}}{\sqrt{n}}$. $\to 1$ $\therefore$ B

$f(x) = \sqrt{x^2+1}$

$x \geq 1$.

$f'(x) = \frac{2x}{2\sqrt{x^2+1}}$

$x \geq 1$

$f(x) = \frac{x^2}{\sqrt{x^3+1}}$, $x \geq 1$

$$f'(x) = \frac{\sqrt{x^3+1}\cdot 2x - x^2 \cdot \frac{3x^2}{2\sqrt{x^3+1}}}{(x^3+1)}$$

$$= \frac{2(x^3+1)\cdot 2x - 3x^4}{2\sqrt{x^3+1}\,(x^3+1)}, \quad x \geq 1$$

$$= \frac{x^4+4x}{2\sqrt{x^3+1}\,(x^3+1)}, \quad x \geq 1$$

$\therefore f'(x) > 0 \Rightarrow f$ is increasing $\Rightarrow$ Seq is increasing

$\Rightarrow$ Seq. is monotone.

State whether the sequence converges and, if it does, find the limit.

$2^n. \to \infty$ dne	$\frac{2}{n}. \to 0$	$\frac{(2n+1)^2}{(3n-1)^2}. \to \left(\frac{2}{3}\right)^2 = \frac{4}{9}$	$\ln\left(\frac{2n}{n+1}\right). \to \ln(2)$
$\frac{(-1)^n}{n}. \to 0$	$\sqrt{n}. \to \infty$ dne	$\frac{n^2}{\sqrt{2n^4+1}}. \to \frac{1}{\sqrt{2}}$	$\frac{n^4-1}{n^4+n-6}. \to 1$
$\frac{n-1}{n}. \to 1$	$\frac{n+(-1)^n}{n}. \to 1$	$\cos n\pi. = (-1)^n$ dne	$\frac{n^5}{17n^4+12}. \to \infty$ dne
$\frac{n+1}{n^2}. \to 0$	$\sin\frac{\pi}{2n}. \to 0$	$e^{1/\sqrt{n}}. \to 1$	$\sqrt{4-\frac{1}{n}}. \to 2$
$\frac{2^n}{4^n+1}. \to 0$	$\frac{n^2}{n+1}. \to \infty$ dne	$\ln n - \ln(n+1).$ $\ln\left(\frac{n}{n+1}\right) \to 0$	$\frac{2^n-1}{2^n}. \to 1$
$(-1)^n\sqrt{n}.$ dne	$\frac{4n}{\sqrt{n^2+1}}. \to 4$	$\frac{\sqrt{n+1}}{2\sqrt{n}}.$	$\frac{1}{n} - \frac{1}{n+1}.$
$\left(-\frac{1}{2}\right)^n \to 0$	$\frac{4^n}{2^n+10^6}. \to \infty$	$\left(1+\frac{1}{n}\right)^{2n}.$	$\left(1+\frac{1}{n}\right)^{n/2}.$ $e^{1/2}$
$\tan\frac{n\pi}{4n+1}.$	$\frac{10^{10}\sqrt{n}}{n+1}.$	$\frac{2^n}{n^2}.$	$2\ln 3n - \ln(n^2+1).$

$$\left(1+\frac{a}{n}\right)^n \to e^a$$

$$\left(1+\frac{1}{n}\right)^{2n} = \left[\left(1+\frac{1}{n}\right)^n\right]^2$$

$$\to e^2$$

$$2\ln(3n) - \ln(n^2+1)$$

$$= \ln(9n^2) - \ln(n^2+1)$$

$$= \ln\left(\frac{9n^2}{n^2+1}\right) \to \ln(9).$$