Test 4 Review
Spring 2012

Topics

Infinite Series:
- Convergence, divergence, absolute convergence, conditional convergence.
- Alternating series and alternating series test.
- Convergence tests for series with nonnegative terms - integral test, comparison test, limit comparison test, ratio test, root test.
- Special series (p-series, geometric series).

L'Hospital's Rule:
- Indeterminant forms.
- Applying the theorem.

Improper Integrals:
- Identification.
- Computation using proper notation.

Taylor Polynomial Approximation:
- Formula for Taylor polynomials.
- Taylor polynomials for simple functions.
- Error estimation and prediction of n to satisfy an error bound.

Practice Questions

Example: Give the value of $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{4^n}$

$\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r|<1$

$\boxed{\displaystyle\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r} \text{ if } |r|<1}$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{4^n}$$

$$= \sum_{n=2}^{\infty} \left(-\frac{1}{4}\right)^n \quad r$$

$$= \frac{\left(-\frac{1}{4}\right)^2}{1-\left(-\frac{1}{4}\right)} = \frac{1/16}{5/4} = \frac{1}{20}$$

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

$\sum \frac{1}{n^p}$

converges if $p>1$

diverges if $p \le 1$.

p-series

$p = \frac{3}{4} \le 1$

$\Rightarrow$ diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

2^n grows faster than n^3

$\therefore$ terms $\not\to 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{n-(n+1)}{n^2+n}$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n^2+n}$$

$$= -\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+n}}$$

use CT or LCT with $\sum \frac{1}{n^2}$ to show this converges

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{9^n}{n!}$$

we "know" this converges b.c. $n!$ grows much faster than 9^n.

Verify: ratio test

$$\lim_{n\to\infty} \frac{9^{n+1}/(n+1)!}{9^n/n!} = \lim_{n\to\infty} \frac{9^{n+1}}{(n+1)!}\cdot\frac{n!}{9^n}$$

$$= \lim_{n\to\infty} \frac{9}{n+1}$$

$$= 0 < 1$$

$\therefore$ the series converges

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \cos(\pi n) = \sum_{n=1}^{\infty} (-1)^n$$

terms $= (-1)^n$

terms $\not\to 0$ as $n \to \infty$

$\Rightarrow$ series diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

p - series with $p = \frac{1}{2}$

Since $\frac{1}{2} \le 1$

the series diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{3n^3+1}$$

Alternating Series

Note: $\lim_{n\to\infty} \frac{n^2}{3n^3+1} = 0$

AND $\frac{n^2}{3n^3+1}$ decrease

for large n.

$\therefore$ by A.S.T., the series converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n = 3 \underbrace{\sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^n}_{\text{geom. series}}$$

$r = -\frac{1}{2}$

$\left|-\frac{1}{2}\right| < 1$

$\Rightarrow$ converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the integral test

$f(x) = \frac{1}{x(\ln(x))^2}$, $x \ge 2$

Note f is decreasing.

Also $\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx$

$= \lim_{a\to\infty} \int_2^a \frac{1}{x(\ln(x))^2}\,dx$

$= \lim_{a\to\infty} -(\ln(x))^{-1} \Big|_2^a$

$= \lim_{a\to\infty} \left[-\frac{1}{\ln(a)} + \frac{1}{\ln(2)}\right]$

$= \frac{1}{\ln(2)}$ Finite.

ex. $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

similar.

you do it.

Diverges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} ne^{-n^3} = \sum_{n=1}^{\infty} \frac{n}{e^{n^3}}$$

we expected convergence since e^{n^3} grows much faster than n.

verify: root test

$$\lim_{n\to\infty} \left(\frac{n}{e^{n^3}}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n} \to 1}{e^{n^2} \to \infty}$$

$$= 0 < 1$$

∴ series converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{n+1-1}{n+1}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right)^n$$

$$\underbrace{\lim_{n\to\infty} \left(1 - \frac{1}{\boxed{n+1}}\right)^{\boxed{n}}}_{\text{terms}} = \lim_{n\to\infty} \left[\underbrace{\left(1 - \frac{1}{n+1}\right)^{n+1}}_{\to e^{-1}}\right]^{\frac{n}{n+1} \to 1}$$

$$= \frac{1}{e} \neq 0$$

Series diverges since terms $\not\to 0$.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$$

L.C.T. or C.T. with

$$\sum \frac{1}{n^3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3+1} \ \underline{\text{conv.}}$$

↑ known convergent p-series

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n$$ Converges

Geom. Series $r=\frac{2}{9}$

$\left|\frac{2}{9}\right|<1$

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=2}^{\infty}\frac{10n^2+n-2}{2n^6+7n-1}$$

L.C.T. with $\sum \frac{1}{n^4}$ will give conv. since $\sum \frac{1}{n^4}$ is a known conv. p-series.

Work: $\lim_{n\to\infty} \dfrac{\frac{10n^2+n-2}{2n^6+7n-1}}{\frac{1}{n^4}}$

$= \lim_{n\to\infty} \dfrac{\boxed{10n^6}+n^5-2n^4}{\boxed{2n^6}+7n-1}$

$= 5 < \infty$

$\therefore$ Our Series converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$$

Since 2^n grows much faster than n^2, we expect conv.

verify: Root test

$\lim_{n\to\infty}\left(\frac{n^2}{2^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{2/n}}{2}$ $\nearrow 1$

$=\frac{1}{2}<1$

$\therefore$ Series converges.

Example: Determine whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty}\frac{n^2+3n-2}{\sqrt{4n^9+n-1}}$$

Note: $\sqrt{4n^9+n-1}$ behaves like $2n^{9/2}$

$\frac{n^2}{2n^{9/2}}=\frac{1}{2n^{5/2}}$ $\quad \frac{5}{2}>1.$

L.C.T. with $\sum\frac{1}{n^{5/2}}$, a known conv. p-series.

$\Rightarrow$ our series conv.

Example: Determine whether the series ~~converges absolutely~~, converges conditionally, or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$$

ABS: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$

L.C.T. with $\sum \frac{1}{n^{1/2}}$ diverges

NO

Cond: Alt. Series

Note: $\frac{\sqrt{n}}{n+3} \to 0$ as $n \to \infty$

Also $\frac{\sqrt{n}}{n+3}$ decreases

$\Rightarrow$ the series conv. by A.S.T.

$\therefore$ The series conv. conditionally.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

ABS: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Known conv. p-series

$p=2>1$.

$\therefore$ ABS conv.

Example: Determine whether the series ~~converges ab~~solutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$$

ABS $\sum \frac{4n}{3n^2+2n+1}$

L.C.T. with $\sum \frac{1}{n}$ diverges

NO

Cond. Alt. Series

Note: $\frac{4n}{3n^2+2n+1} \to 0$

AND $\frac{4n}{3n^2+2n+1}$ decreases for large n

b.c. $3n^2+2n+1$ eventually grows much faster than $4n$.

$\therefore$ Series conv.

$\therefore$ Series conv. Cond.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$$

ABS $\sum_{n=0}^{\infty} \frac{3}{\sqrt{3n^2+2n+1}}$

L.C.T. with $\sum \frac{1}{n}$ diverges

NO

Cond. Alt. Series.

... done

Series conv.

$\therefore$ Cond. Conv.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$$

Note: Terms $\not\to 0$.

$\therefore$ Series diverges.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \left(4(-1)^n \boxed{\left(\frac{n}{n+3} \right)^n} \right)$$

$\to e^{-1}$ as $n \to \infty$

Terms $\not\to 0$

$\Rightarrow$ Series diverges.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \left(\frac{2(-1)^n \arctan n}{3+n^2+n^3} \right)$$

ABS $\displaystyle\sum_{n=0}^{\infty} \frac{2\arctan(n)}{3+n^2+n^3}$

L.C.T. with $\sum \frac{\pi}{n^3}$.

$\therefore$ ABS Conv.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n 3^n}{4^n+3n} \right)$$

ABS: $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n+3n}$

L.C.T. with $\sum \left(\frac{3}{4}\right)^n$, a known conv. geom series.

$\therefore$ ABS conv.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3}{(n+2)\ln(n+2)}\right)$$

ABS? NO $\sum_{n=0}^{\infty} \frac{1}{(n+2)\ln(n+2)}$

Int. test.

Cond? Yes. A.S.T.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty}\frac{(-1)^n n!}{(n+1)!} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1}$$

ABS: $\sum \frac{1}{n+1}$ NO

Cond: Alt Series $\frac{1}{n+1} \to 0$

Yes $\frac{1}{n+1}$ decreases.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{3n+2}$$

ABS: $\sum \frac{1}{3n+2}$ NO

Cond: Alt. Series $\frac{1}{3n+2} \to 0$

Yes $\frac{1}{3n+2}$ decrease

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=0}^{\infty}\frac{(-1)^n 10n^2}{3^n}$$

ABS $\sum_{n=0}^{\infty} \frac{10n^2}{3^n}$.

use root or ratio test.

Yes

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{n!}$$

ABS $\sum_{n=2}^{\infty} \frac{3^n}{n!}$

ratio test $\Rightarrow$ Yes

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+3n+2}$$

ABS $\sum_{n=2}^{\infty} \frac{1}{n^2+3n+2}$

Yes

L.C.T. or C.T. with $\sum \frac{1}{n^2}$

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty} \frac{\overbrace{\boxed{\cos(\pi n)}}^{=(-1)^n} n^n}{n!}$$

n^n grows faster than $n!$

$\therefore$ terms $\not\to 0$

$\Rightarrow$ series diverges.

Example: Determine whether the series converges absolutely, converges conditionally, or diverges. Show your work.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

done earlier

Integral test.

Converges

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{n\to\infty} \frac{\ln(n+4)}{n+2} = 0$$

"$\frac{\infty}{\infty}$"

L'H: $\lim_{n\to\infty} \frac{\frac{1}{n+4}}{1} = 0$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{n\to\infty} (3n)^{\frac{2}{n}} = \lim_{n\to\infty} (3n)^{2/n} = 1.$$

"∞^0"

$$(3n)^{2/n} = e^{\ln((3n)^{2/n})}$$

$$= e^{\frac{2}{n}\ln(3n)} = e^{\boxed{\frac{2\ln(3n)}{2n}}} \to e^0 = 1$$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{2n} = \lim_{n\to\infty} \left[\left(1+\frac{3}{n}\right)^n\right]^2$$

$$= (e^3)^2 = e^6.$$

"1^∞"

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0} \frac{x-\sin(2x)}{x+\sin(2x)} = -\frac{1}{3}$$

"$\frac{0}{0}$"

L-H: $\lim_{x\to 0} \frac{1-2\cos(2x)}{1+2\cos(2x)} = -\frac{1}{3}$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0}\frac{e^{x^2}-1}{2x^2} = \frac{1}{2}$$

"$\frac{0}{0}$"

L-H: $\lim_{x\to 0}\frac{2xe^{x^2}}{4x} = \lim_{x\to 0}\frac{e^{x^2}}{2}$

$= \frac{1}{2}$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0+}\left(\frac{1}{x}\right)^x = 1$$

"∞^0"

$$\left(\frac{1}{x}\right)^x = e^{\ln\left(\left(\frac{1}{x}\right)^x\right)} = e^{x\ln\left(\frac{1}{x}\right)} = e^{\frac{\ln\left(\frac{1}{x}\right)}{1/x}} = e^{\boxed{\frac{\ln(u)}{u}}\to 0}$$

$u\to\infty$

$\to e^0 = 1$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0}\frac{3e^{x/3}-(3+x)}{x^2} = \frac{1}{6}$$

"$\frac{0}{0}$"

L-H: $\lim_{x\to 0}\frac{e^{x/3}-1}{2x} = \frac{1}{6}$

"$\frac{0}{0}$" L-H: $\lim_{x\to 0}\frac{\frac{1}{3}e^{x/3}}{2} = \frac{1}{6}$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to\infty}\frac{x^2}{\ln x} = \infty$$

"$\frac{\infty}{\infty}$"

L-H: $\lim_{x\to\infty}\frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0}\frac{1+x-e^{x}}{x(e^{x}-1)} = -\frac{1}{2}$$

"$\frac{0}{0}$"

L-H: $\lim_{x\to 0}\frac{1-e^{x}}{e^{x}-1+xe^{x}} = -\frac{1}{2}$

"$\frac{0}{0}$" L-H: $\lim_{x\to 0}\frac{-e^{x}}{e^{x}+e^{x}+xe^{x}} = -\frac{1}{2}$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to 0}\frac{\arctan(4x)}{x} = 4$$

"$\frac{0}{0}$"

L-H: $\lim_{x\to 0}\frac{\frac{1}{1+(4x)^2}\cdot 4}{1} = 4$

Example: Determine whether the limit is in indeterminant form. Then compute the limit.

$$\lim_{x\to\infty}\frac{\sin(2x)}{3x}$$

bounce btwn -1 and 1

∞

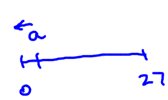
$= 0$

Example: Evaluate the limit using proper notation.

$$\int_0^{27} x^{-2/3}\,dx = \int_0^{27}\frac{1}{x^{2/3}}\,dx$$

$\frac{1}{x^{2/3}}$ is undefined at $x=0$.

$\therefore$ Improper.



$$= \lim_{a\to 0^+}\int_a^{27} x^{-2/3}\,dx$$

$$= \lim_{a\to 0^+} 3x^{1/3}\Big|_a^{27}$$

$$= \lim_{a\to 0^+}\left[9-3a^{1/3}\right] = 9.$$

Example: Evaluate the limit using proper notation.

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$$

Note: $\frac{1}{\sqrt{4-x}}$ is undefined at $x=4$

$$= \lim_{a\to 4^-}\int_0^a \frac{1}{\sqrt{4-x}}\,dx$$

$$= \lim_{a\to 4^-}\int_0^a (4-x)^{-\frac{1}{2}}\,dx$$

$$= \lim_{a\to 4^-} -2(4-x)^{1/2}\Big|_0^a$$

$$= \lim_{a\to 4^-}\left[-2\sqrt{4-a}+2\cdot 2\right]$$

$$= 4.$$

Example: Evaluate the limit using proper notation.

$$\int_{-2}^0 \frac{1}{x+1}\,dx$$

$\frac{1}{x+1}$ is undefined at -1.

$\Rightarrow$ Improper.

$$= \lim_{a\to -1^-}\int_{-2}^a \frac{1}{x+1}\,dx + \lim_{b\to -1^+}\int_b^0 \frac{1}{x+1}\,dx$$

$\ldots = $ undefined

Example: Evaluate the limit using proper notation.

$$\int_1^\infty \frac{1}{x^2+1}\,dx$$

Having ∞ in the limits of integration makes this improper.

$$= \lim_{a\to\infty}\int_1^a \frac{1}{x^2+1}\,dx$$

$$= \lim_{a\to\infty} \arctan(x)\Big|_1^a$$

$$= \lim_{a\to\infty}\left[\arctan(a)-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

Example: Evaluate the limit using proper notation.

$$\int_1^4 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_1^4 = \ln(5)-\ln(2)$$

Not improper b/c $\frac{1}{x+1}$ is continuous on $[1,4]$.

Example: Give the 4th degree Taylor polynomial centered at 0 for $\sin(x)$

$$x - \frac{x^3}{6}.$$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

$$x - \frac{x^3}{6}.$$

Example: Give the 4th degree Taylor polynomial centered at 0 for $\cos(x)$

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Example: Give the 4th degree Taylor polynomial centered at 0 for $\exp(x)$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Example: Give the 4th degree Taylor polynomial centered at 0 for $\ln(x+1)$

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

$f(x) = \ln(x+1)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{x+1}$ $\quad f'(0) = 1$

$f''(x) = \frac{-1}{(x+1)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{2}{(x+1)^3}$ $\quad f'''(0) = 2$

$f^{(4)}(x) = \frac{-6}{(x+1)^4}$ $\quad f^{(4)}(0) = -6$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Example: Give the 4th degree Taylor polynomial centered at 0 for $\cos(2x)$

$f(x) = \cos(2x)$ $\quad f(0) = 1$

$f'(x) = -2\sin(2x)$ $\quad f'(0) = 0$

$f''(x) = -4\cos(2x)$ $\quad f''(0) = -4$

$f'''(x) = 8\sin(2x)$ $\quad f'''(0) = 0$

$f^{(4)}(x) = 16\cos(2x)$ $\quad f^{(4)}(0) = 16$

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 .$$

Example: Give the 6th degree Taylor polynomial centered at 0 for $\sin(x^2)$

Work!

$$\boxed{x^2 - \frac{x^6}{6}}$$

Example: Write the following in powers of $(x + 1)$: $3x^3 - x^2 + 7x - 6$

$(x - \underline{-1})$

center.

$f(x) = 3x^3 - x^2 + 7x - 6$ $\quad f(-1) = -17$

$f'(x) = 9x^2 - 2x + 7$ $\quad f'(-1) = 18$

$f''(x) = 18x - 2$ $\quad f''(-1) = -20$

$f'''(x) = 18$ $\quad f'''(-1) = 18$

$$f(-1) + f'(-1)(x - -1) + \frac{f''(-1)}{2}(x - -1)^2 + \frac{f'''(-1)}{6}(x - -1)^3$$

$$= -17 + 18(x+1) - 10(x+1)^2 + 3(x+1)^3$$

Example: $f(0) = 1, f'(0) = 2, f''(0) = -4$ and $f'''(0) = 1/2$. Give the 3rd degree Taylor polynomial centered at 0.

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$= 1 + 2x - 2x^2 + \frac{1}{12}x^3$$

Example: $f(-1) = -2$, $f'(-1) = 3$, $f''(-1) = 4$ and $f'''(-1) = 3/2$. Give the 3rd degree Taylor polynomial centered at -1.

$$f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2}(x+1)^2 + \frac{f'''(-1)}{6}(x+1)^3$$

$$= -2 + 3(x+1) + 2(x+1)^2 + \frac{1}{4}(x+1)^3$$

Example: Give the smallest value of n so that the n^{th} degree Taylor polynomial centered at 0 approximates exp(-2) within 10^{-1}.

$f(x) = e^x$

unknown.

error is given

we want

$$|f(-2) - P_{n,0}(-2)| \le \frac{1}{10}$$

we know

$$|f(-2) - P_{n,0}(-2)| \le \frac{M^{=1}}{(n+1)!}|-2-0|^{n+1} \le \frac{1}{10}$$

Force

where $M \ge |f^{(n+1)}(x)| = |e^x| = e^x$

$-2 \le x \le 0$

Note: $f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$

$\therefore$ M=1 works

largest on [-2,0] at x=0.

Let's find n so that $\frac{2^{n+1}}{(n+1)!} \le \frac{1}{10}$

	n	$\frac{2^{n+1}}{(n+1)!}$
No	4	$\frac{32}{120}$
Yes	5	$\frac{64}{720} < \frac{1}{10}$
	6	

$2^5 = 32$ $\quad 2^6 = 64$

$5! = 120$ $\quad 6! = 720$

$n = 5$

Example: Give the smallest value of n so that the n^{th} degree Taylor polynomial centered at $\pi/3$ approximates $\cos(70^o)$ within 10^{-3}.