# Online Math 3321
# Summer 2012

http://www.math.uh.edu/~jmorgan/Math3321online

**Today...**

* **Differential Equations** *
- Definitions
- Examples

"Initial Value Problem"

**Initial Value Problems**
- Definitions
- Examples

**Finding Solutions to Specific Types of ODEs and IVPs**
- Separable Equations
- Linear Equations

"Ordinary Differential Equations"

An equation for an unknown function involving one or more derivatives of the function.
This equation could include the independent variable and the function itself.

**Definitions**

- Differential Equation and the Order of a Differential Equation

order of the highest derivative appearing in the equation.

- Ordinary Differential Equations vs Partial Differential Equations

only one variable.

diff eq for functions of more than one variable.
Consequently, the derivatives that appear are partial derivatives.

- Solution of a Differential Equation

A function that satisfies the diff eq.

## Examples of Differential Equations

1. $u'(t) - \cos(t)\left(u(t)^2 + 1\right) = \sin(t)$

$u(t) \equiv$ unknown function.

$t \equiv$ independent variable

$1^{\underline{st}}$ order diff. eq.

Ordinary diff eq.

2. $y'' + \dfrac{x}{y+1} = 2xy'$

$y \equiv$ unknown function

$x \equiv$ ind. variable.

$2^{\underline{nd}}$ order ODE

3. $w' = 2w$

$w \equiv$ unknown function

we don't know the name of the ind. variable. – anything will do.

1st order ODE

4. $u_{xx} + 2u_y = \sin(x+y)$

Note: $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ , $u_y = \dfrac{\partial u}{\partial y}$

Here $u \equiv u(x,y)$

$x, y \equiv$ ind. variables.

2nd order PDE (Partial Diff. Eq.)

**EMCF01b**

2. $y''' - 3y'' + 3\sin(x) = 2 - y$

a. is a first order differential equation

b. is not a differential equation

c. is a second order differential equation

d. is a third order differential equation

1. $y' = x$ First order ODE

Solutions: $\frac{1}{2}x^2$, $\frac{1}{2}x^2+1$, $\frac{1}{2}x^2 - \frac{3}{2}$, $\underbrace{\frac{1}{2}x^2 + C}$

Infinitely many solutions.

C is an arbitrary constant. "General solution"

2. $y' = y$ (Let's use $t$ as ind. variable.)

Solutions: $e^t$, $Ce^t$

C is an arbitrary constant.

*** Examples of Solutions to Differential Equations ***

3. $u_{xx} + u_y = x + y$

Solutions: $u = \frac{x^3}{6} + \frac{y^2}{2} + C$

arbitrary constant

Check it!

How?

Plug it in. $u_x = \frac{x^2}{2}$, $u_{xx} = x$

$u_y = y$

$\therefore \; u_{xx} + u_y = x + y$ ✓

Other sol'ns: $u = \frac{x^3}{6} + \frac{y^2}{2} + C_1 x + C_2$

Here $C_1$ and $C_2$ are arbitrary constants.

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3. The function $2\sin(x) + x$ solves the differential equation

$\times$ a. $y'' - y = x$

$\times$ b. $y'' + y = -x$

$\times$ c. $y'' - 2y = -2x$

$\times$ d. $y'' + 2y = 2x$

e. None of these.

$* \; y = 2\sin(x) + x$

$y' = 2\cos(x) + 1$

$\# \; y'' = -2\sin(x)$

**Remark:** Differential equations typically have infinitely many solutions. We need to give some other information to pinpoint a specific solution.

The *general solution* to a differential equation is an expression (often involving arbitrary constants) that can be used to generate every solution of the differential equation.

**Definitions**

- Initial Value Problem:

a differential equation, coupled with initial data...

More specifically, an initial value problem is a $k^{th}$ order differential equation along with the values of $y, y', \ldots, y^{(k-1)}$ at a given value of the independent variable.

Diff Eq
+
Initial Data

- Solution of an Initial Value Problem:

a function that solves the differential equation AND satisfies the initial data.

ex. $y' = y \quad , \quad y(0) = 3$

i.e. $y = 3$ when $t = 0$.

1st order IVP.

$$y = Ce^{t} \rightarrow 3 = Ce^{0} = C$$

$$\Rightarrow \boxed{y = 3e^{t}}$$

$$\underbrace{y'' - 3xy' = \sin(x)}_{\text{Second order ODE}}, \quad \underbrace{y(1) = 2, \ y'(1) = -3}_{\text{Initial Data.}}$$

## * Examples of Initial Value Problems *

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4. $y'' + 2y = \cos(x), y(0) = 3, y'(0) = 2$

a. is a first order differential equation

b. is not an initial value problem

c. is a second order initial value problem

d. is a first order initial value problem

**Remark:** Initial value problems typically only have 1 solution.

**Theorem:** If $x_0$ and $y_0$ are given real numbers, and $f(x,y)$ is continuously differentiable at every point $(x,y)$ near $(x_0,y_0)$, then there is an open interval containing $x_0$ on which there is exactly one solution to the initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

1st order IVP

derivative of f is a continuous function.

i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

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5. The function $2\sin(x)$ solves the initial value problem

a. $y'' - y = 0,\ y(0) = 2,\ y'(0) = 0$

b. $y'' + y = 0,\ y(0) = 0,\ y'(0) = 2$

c. $y'' - 2y = 0,\ y(0) = 0,\ y'(0) = 2$

d. $y'' + 2y = 0,\ y(0) = 2,\ y'(0) = 0$

$y = 2\sin(x)$

Plug it in!

$y' = 2\cos(x)$

$y'' = -2\sin(x)$

$\therefore\ y'' + y = 0$

$y(0) = 2\sin(0) = 0$

$y'(0) = 2\cos(0) = 2$

ex. $y' = \sin(x^2)$ , $y' = y^2 + t$

**The Truth:** There are many differential equations for which it is IMPOSSIBLE to "write a formula" for a solution.

This does not mean there is not a solution. It just means we can't write the formula for the solution.

Two types that we can solve are

1. **First Order Separable Differential Equations**

and

2. **First Order Linear Differential Equations**

## First Order Separable Differential Equations

$$(*) \longrightarrow \frac{dy}{dx} = f(x)\,g(y)$$

**Examples:** $\dfrac{dy}{dx} = -xy,\qquad \dfrac{dy}{dx} = \dfrac{2x}{y+1},\qquad \dfrac{dz}{dt} = \dfrac{2z\cos(t)}{\sin(t)+3}$

$$(-x)(y)$$

$$(2x)\left(\frac{1}{y+1}\right)$$

$$\left(\frac{2\cos(t)}{\sin(t)+3}\right)\cdot(z)$$

**Solving Separable Differential Equations**
(finding the general solution)

$$\frac{dy}{dx} = f(x)g(y)$$

1. Separate $\longrightarrow$ $\dfrac{dy}{g(y)} = f(x)\,dx$

2. Integrate $\longrightarrow$ $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$ *

3. Solve for the unknown function (if possible) to obtain the form of the general solution.

Could be a sticking point b/c the integrals could be impossible to compute.

1st order IVP $\longleftrightarrow$ $\frac{dy}{dx} = 2e^{2x+y}$, $y(0) = 3$

**Example:** Find the solution to $\frac{dy}{dx} = 2e^{2x+y}$, $y(0) = 3$

Q: Is $\frac{dy}{dx} = 2e^{2x+y}$ separable? yes

$$\frac{dy}{dx} = (2e^{2x})(e^{y})$$

I. Find the general sol'n.

II. Use the initial data $y(0) = 3$ to pinpoint the solution.

1. Separate. $\frac{dy}{e^{y}} = 2e^{2x}\,dx$

$$e^{-y}\,dy = 2e^{2x}\,dx$$

2. Integrate $\int e^{-y}\,dy = \int 2e^{2x}\,dx$

$$\boxed{-e^{-y} = e^{2x} + C}$$

**Note: There will be a constant of integration on both sides, and we will combine them as one constant on the right hand side.**

implicitly gives $y$.

$$\boxed{-e^{-y} = e^{2x} + C}$$

implicitly gives $y$.

Let's satisfy the initial data.

$y(0) = 3$. $\longleftrightarrow$ $y = 3$ when $x = 0$.

$$-e^{-3} = e^{2\cdot 0} + C$$

$$-1 - e^{-3} = C \Rightarrow -e^{-y} = e^{2x} - 1 - e^{-3}$$

implicit representation of the sol'n $y$.

Note: In this case, we can get $y$ explicitly. (solve for $y$).

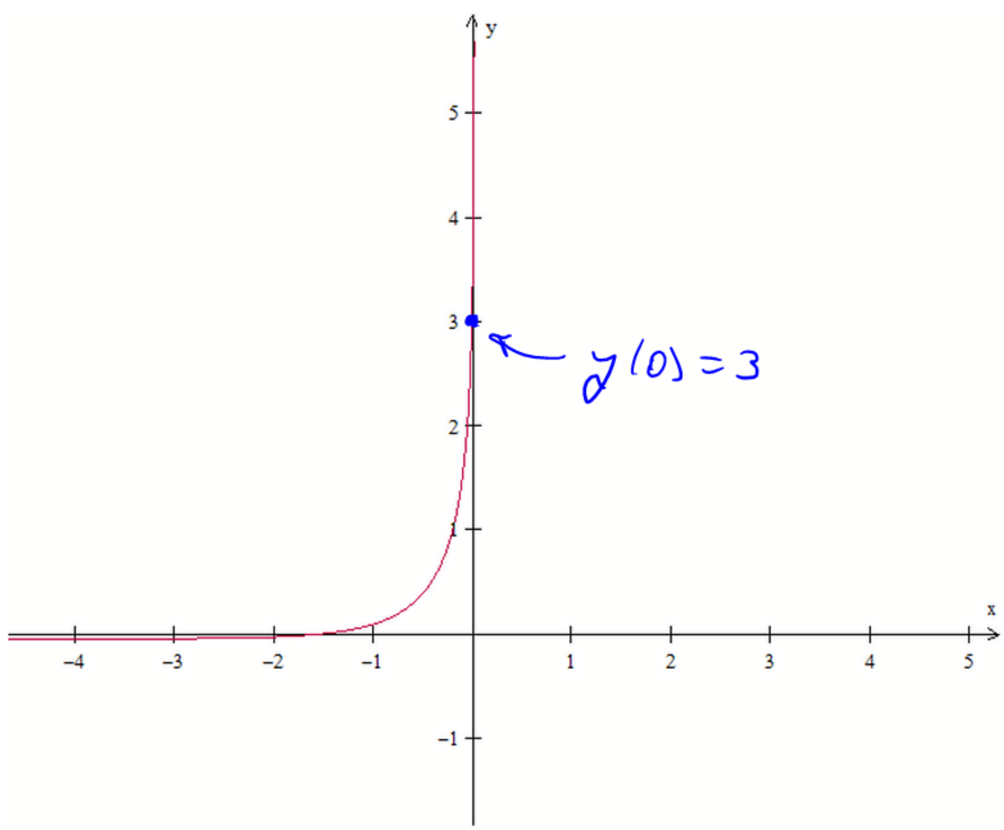
$$e^{-y} = -e^{2x} + 1 + e^{-3} \Rightarrow -y = \ln\left(-e^{2x} + 1 + e^{-3}\right)$$

$$\Rightarrow y = -\ln\left(-e^{2x} + 1 + e^{-3}\right)$$

Let's plot this.

**winplot** - Free Software ← For PC.

http://math.exeter.edu/rparris/winplot.html



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6. Which of the following differential equations is/are NOT separable?

i. $y' = -2y\,e^{-3x}$ Separable

ii. $y' = xy - e^{2x}$ Not

iii. $y' = -x/(y + x)$ Not

a. i and ii
b. i and iii
c. ii and iii
d. i only
e. None of the above.

**Important Special Case**

(of a separable diff eq.)

$$\frac{dy}{dx} = k\,y$$

k is a real #.

$$\frac{dy}{y} = k\,dx \Rightarrow \int \frac{dy}{y} = \int k\,dx$$

$$\Rightarrow \ln|y| = kx + C_1 \Rightarrow |y| = e^{kx+C_1}$$

$$\Rightarrow |y| = e^{kx} \cdot \underline{e^{C_1}} = C e^{kx}$$

constant

different from $C_1$

$$\Rightarrow y = \underbrace{\pm C}_{\text{constant.}} e^{kx}$$

---

$$\therefore \quad \frac{dy}{dx} = ky \iff y = C e^{kx}$$

constant

**In General...**

$$\frac{dy}{dx} = k\,y \quad \Leftrightarrow \quad y = C\,e^{kx}$$

**Note:** Other independent/dependent variable names are also possible.

$$\frac{du}{dt} = -4\,u \quad \Leftrightarrow \quad u = C\,e^{-4t}$$

$$u'(t) = 2\,u(t) \quad \Leftrightarrow \quad u(t) = C\,e^{2t}$$

$$w'(z) = -3\,w(z) \quad \Leftrightarrow \quad w(z) = C\,e^{-3z}$$

$y' = 2y \Leftrightarrow y = Ce^{2x}$ | $y' = -3y \Leftrightarrow y = Ce^{-3x}$

**EMCF01b**

7. Give the value of the solution to $P'(x) = -2P(x),\ P(0) = 3$ at $x = 1$.

$a.\ -2e^{3}$

$b.\ 2e^{-3}$

$c.\ 3e^{-2}$ ✓

$d.\ 3e^{2}$

$e.$ None of these.

Find $P(x)$. Then give $P(1)$.

$$P(x) = C_1 e^{-2x}$$

use $P(0) = 3$

$$3 = C_1 e^{-2\cdot 0} \Rightarrow C_1 = 3$$

$$\Rightarrow P(x) = 3e^{-2x} \Rightarrow P(1) = 3e^{-2}.$$

## First Order Linear Differential Equations

(*) $\longleftrightarrow$ $\dfrac{dy}{dx} + p(x)y = f(x)$

$p(x)$, $f(x)$ are given.

**Illustrative Example:** Find the general solution to $\underbrace{\dfrac{dy}{dx} = -2y + e^{2x}}$

$$\frac{dy}{dx} + \underbrace{2}_{p(x)} y = \underbrace{e^{2x}}_{f(x)}$$

integrating factor

Solution Idea: Find a function $h(x)$ so that

$$\boxed{h(x)\frac{dy}{dx} + 2h(x)y} = h(x)e^{2x}$$

results in the left hand side being

$$\frac{d}{dx}\left(h(x)y\right) = \boxed{h(x)\frac{dy}{dx} + h'(x)y}$$

Need $h'(x) = 2h(x)$.

Solution Idea: Find a function $h(x)$ so that

$$\boxed{h(x)\frac{dy}{dx} + 2h(x)y} = h(x)e^{2x}$$

Any $h(x)$ will do except $h(x)=0$

results in the left hand side being

$$\frac{d}{dx}\left(h(x)y\right) = \boxed{h(x)\frac{dy}{dx} + h'(x)y}$$

Need $h'(x) = 2h(x)$.

↳ use $h(x) = e^{2x}$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}\cdot e^{2x} = e^{4x}$$

Integrate:

$$e^{2x}y = \int e^{4x}\,dx$$

$$e^{2x}y = \frac{1}{4}e^{4x} + C$$

$$\boxed{y = \frac{1}{4}e^{2x} + Ce^{-2x}}$$

**Solution Process for First Order Linear Differential Equations** (how to find the general solution)

$$\frac{dy}{dx} + p(x)y = f(x)$$

**Step 1:** Make the left hand side look like the derivative of a product by multiplying both sides by a special function $h(x)$.

$$\boxed{h(x)\frac{dy}{dx} + \underline{h(x)p(x)}\,y} = h(x)f(x)$$

where h(x) is chosen so that the left hand side is

$$\frac{d}{dx}\left(h(x)y\right) = \boxed{h(x)\frac{dy}{dx} + \underline{h'(x)}\,y}$$

i.e. $h'(x) = h(x)p(x)$

that is $\frac{dh}{dx} = p(x)h$

separable!

i.e. $h'(x) = h(x)\,p(x)$

that is $\underbrace{\dfrac{dh}{dx} = p(x)\,h}$

separable!

$$\frac{dh}{h} = p(x)\,dx \,. \qquad \int \frac{dh}{h} = \int p(x)\,dx$$

(Note: we don't need the most general $h(x)$. Any nonzero one will do.)

$$\ln(h) = \int p(x)\,dx$$

$$\Rightarrow \quad \boxed{h = e^{\int p(x)\,dx}}$$

Using this, we have

$$\frac{d}{dx}\left(h(x)\,y\right) = h(x)\,f(x)$$

Integrate and solve for $y$.

**Example:** Solve the initial value problem $\underbrace{\frac{dy}{dx}+xy=2xe^{-x^2}}$, $y(0)=-1$

1st order linear

$p(x)=x$, $f(x)=2xe^{-x^2}$

① Get an integrating factor.

$$h(x)=e^{\int p(x)dx}=e^{\int x\,dx}=e^{\frac{1}{2}x^2}$$

we do not need the most general

② mult both sides by $h(x)$.

$$\underbrace{e^{\frac{1}{2}x^2}\frac{dy}{dx}+e^{\frac{1}{2}x^2}xy}_{\frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right)}=e^{\frac{1}{2}x^2}2xe^{-x^2}$$

$$\frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right)=2xe^{-\frac{1}{2}x^2}$$

③ Integrate

$$e^{\frac{1}{2}x^2}y=\int 2xe^{-\frac{1}{2}x^2}dx$$

$$e^{\frac{1}{2}x^2} y = \int 2x e^{-\frac{1}{2}x^2} dx$$

$u = -\frac{1}{2}x^2$

$du = -x\,dx$

$$\Rightarrow e^{\frac{1}{2}x^2} y = -2e^{-\frac{1}{2}x^2} + C$$

$$\Rightarrow \boxed{y = -2e^{-x^2} + Ce^{-\frac{1}{2}x^2}}$$

General sol'n to

$$\frac{dy}{dx} + xy = 2xe^{-x^2}$$

Now pinpoint the solution by using the initial data. $y(0) = -1$.

$\hookrightarrow y = -1$ when $x = 0$.

$$-1 = -2e^{-0^2} + Ce^{-\frac{1}{2}0^2}$$

$\Rightarrow C = 1$

$$\Rightarrow \boxed{y = -2e^{-x^2} + e^{-\frac{1}{2}x^2}}$$

Sammy's Question:

$$y' + p(x)\,y = 0$$

$$h(x) = \int p(x)\,dx$$

← an anti-derivative of $p(x)$

One choice

$$\int_a^x p(t)\,dt$$

Recall the fundamental Thm of Calc.

$$\frac{d}{dx}\int_a^x p(t)\,dt = p(x)$$

i.e. $\int_a^x p(t)\,dt$ is an anti-derivative of $p(x)$.