Review, Bernoulli and Homogeneous Equations, Simple Applications and Approximating Solutions

EMCF

"EMCF with mixed format"

Open **EMCF02** on courseware at http://www.casa.uh.edu

Note: Assignment 2 is posted!!

Math 3321 - 15894

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Read the Syllabus

Note: Quizzes are given online on CourseWare. Check the calendar below for due dates.

The online text is available on CourseWare. Purchase an Access Code from the UC Book Store to access the online text, videos, EMCFs and quizzes.

Use this link to access the online Thursday sessions.

Use the Discussion Board on CourseWare to get and give help.

Course Calendar

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
June 2	3 **Assignment 1** Week 1 Video and Lecture Notes from Summer 2012 **Note:** You are not responsible for the listed EMCF questions in these videos.	4 Wolfram Alpha Widgets for Solving and Plotting. **Note:** You will learn to solve differential equations by hand, but it's nice to have a tool for checking your work.	5	6 online live 4-6pm **link to the session** **Notes for the Session, video** **Access Codes are due!!**	7 **Quiz 1 Due 1.1-1.3** **Alternate 1** You can find the answer sheet for these questions by logging in to CourseWare, selecting the EMCF tab, and clicking on "EMCF with Mixed Format" and selecting Alternate01.	8
9	10 **Assignment 2** Week 2: ***From Summer 2012*** Videos: 1 and 2, and Lecture Notes: 1 and 2 **Note:** You are not responsible for the EMCF questions in these videos.	11 **Alternate 1 due** **Assignment 1 due** Log in to CourseWare. Then access the Assignment tab and upload a scanned version of your homework.	12 **Quiz 2 Due 2.1-2.2**	13 online live 4-6pm **link to the session** **Blank Slides**	14 Euler's: **by hand, with Excel, with VBA Excel, with Matlab** Improved Euler's: **by hand, with Excel, with VBA Excel, with Matlab**	15

Last time we discussed: solution

ODE, IVP,

- Definitions
- Examples
- Separable equations
- Linear equations

we can solve these types of first order ODEs provided we can integrate.

Did you try the online solver?

Today:

- Bernoulli and Homogeneous equations
- Applications of linear equations
- Direction fields
- Approximating solutions using Euler's method and Improved Euler's method (see the posted part 2 video from Monday, and the additional demonstration videos)

Review:

First Order Separable Differential Equations

$$\frac{dy}{dx} = f(x)g(y)$$

Process:

1. $\dfrac{dy}{g(y)} = f(x)\,dx$
2. Integrate $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$

Solve for y, if possible.

First Order Linear Differential Equations

$$\frac{dy}{dx} + p(x)y = f(x)$$

Process:

1. Get an integrating factor.

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

any anti-derivative will do.

2. $\underbrace{\mu(x)\dfrac{dy}{dx} + p(x)\mu(x)y}_{\dfrac{d}{dx}(\mu(x)y)} = \mu(x)f(x)$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$$

3. Integrate. $\mu(x)y = \underline{\int \mu(x)f(x)\,dx}$

thrown in a const of integration.

Review (continued): **Special Case** $k \equiv \text{constant}$

$$\underbrace{\frac{dy}{dx} = k\,y}_{\text{why?}} \quad \Leftrightarrow \quad y = C\,e^{kx}$$

ex. $y' = 3y$

$\Rightarrow y = Ce^{3x}$

$y' + (-k)y = 0$

$\mu(x) = \exp\left(\int -k\,dx\right) = e^{-kx}$

$e^{-kx}y' + (-k)e^{-kx}y = 0$

$\frac{d}{dx}\left(e^{-kx}y\right) = 0 \Rightarrow e^{-kx}y = C$

$\Rightarrow \boxed{y = Ce^{kx}}$

$2e^{-2} = \underbrace{0.2706}70566$

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$y = Ce^{-2x}$

$y(0)=2 \Rightarrow 2 = Ce^{-2(0)} = C$

$\Rightarrow \underbrace{y = 2e^{-2x}} \Rightarrow y(1) = 2e^{-2}$

1. Find the solution to the initial value problem $y' = -2y$, $y(0) = 2$. Then evaluate this solution at $x = 1$.

→ set floating point accurate to 4 digits to the right of the decimal.

2. Find the solution to the initial value problem $y' = 3y$, $y(1) = 2$. Then evaluate this solution at $x = 0$.

$2e^{-3} = 0.0995$

$y = Ce^{3x}$

$2 = Ce^{3}$

$\Rightarrow C = 2e^{-3}$

$\Rightarrow y = 2e^{-3}e^{3x} = 2e^{3x-3}$

$\Rightarrow y(0) = 2e^{-3}$

New

Bernoulli Differential Equations - Section 2.3

(use a substitution to create a first order linear differential equation)

$$y' + p(x)y = q(x)y^r, \quad \boxed{r \neq 0,1}$$

Note: $r=0$ gives $y' + p(x)y = q(x)$

$r=1$ gives $y' + (p(x) - q(x))\,y = 0.$

1st order linear

Observation: $\frac{1}{y^r} y' + p(x)\, y^{1-r} = q(x)$

Note: $\frac{d}{dx} y^{1-r} = (1-r)\, y^{-r} \frac{dy}{dx}$

Let $v = y^{1-r} \Rightarrow v' = (1-r)\frac{1}{y^r} y'$

$\Rightarrow \frac{1}{1-r} v' = \frac{1}{y^r} y'$

Substitute: $\frac{1}{1-r} v' + p(x)\, v = q(x)$

$v' + (1-r)p(x)\, v = (1-r)\, q(x)$

First order linear for v.

Find v. Then $v = y^{1-r}$ gives y.

Bernoulli diff eq.

$$y' + p(x)y = q(x)y^r,\ r \neq 0,1$$

Example: Find the general solution to

$$y' + \frac{1}{x}y = 4x^3y^3 \quad \leftarrow \text{Bernoulli}$$

$$\underbrace{\frac{1}{y^3}y'}_{} + \frac{1}{x}\cdot\underbrace{\frac{1}{y^2}}_{} = 4x^3$$

↑ new function.

Set $\boxed{v = y^{-2}} \Rightarrow v' = -2\underbrace{y^{-3}y'}_{}$

$$-\frac{1}{2}v' + \frac{1}{x}v = 4x^3$$

$$\boxed{v' + \frac{-2}{x}v = -8x^3} \leftarrow 1^{st} \text{ order linear in } v.$$

$$\mu(x) = \exp\left(\int -\frac{2}{x}dx\right) = e^{-2\ln|x|} = x^{-2}$$

be careful

$$\underbrace{x^{-2}v' + \frac{-2}{x}x^{-2}v}_{\parallel} = x^{-2}(-8x^3)$$

$$\frac{d}{dx}\left(x^{-2}v\right) = -8x$$

integrate:

$$x^{-2}v = -4x^2 + C$$

$$\Rightarrow \quad v = -4x^4 + Cx^2$$

recall: $v = y^{-2}$

$$\Rightarrow \quad y^{-2} = -4x^4 + Cx^2$$

$$\Rightarrow \quad y^2 = \frac{1}{-4x^4 + Cx^2}$$

$$\Rightarrow \quad y = \pm \frac{1}{\sqrt{-4x^4 + Cx^2}}$$

$$y = \pm \frac{1}{\sqrt{-4x^4 + Cx^2}}$$

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3. Find the unique solution to the ODE from the previous example that satisfies $y(1) = 1$. Then give the value of $y(0.5)$.

y starts out positive. use "+" above.

$$y = \frac{1}{\sqrt{-4x^4 + Cx^2}}$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{\sqrt{-4 + C}} \Rightarrow C = 5.$$

$$\Rightarrow y = \frac{1}{\sqrt{-4x^4 + 5x^2}} \Rightarrow y\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{-\frac{1}{4} + \frac{5}{4}}}$$

$$= 1.$$

New

Homogeneous Differential Equations - Section 2.3

(use a substitution to create a first order separable differential equation)

Special case

$$\boxed{y' = f(x,y)}$$

* where $f(\alpha x, \alpha y) = f(x,y)$ * for all α, x, y.

determines whether the ODE is homogeneous.

Idea : Create a new function v via $y = xv$.

let's see it : Substitute $y = xv$.

$$\Rightarrow \boxed{y' = xv' + v}$$

$$\Rightarrow xv' + v = f(x, xv) = f(x \cdot 1, x \cdot v) = f(1, v)$$

$$\Rightarrow xv' + v = \boxed{f(1,v)}$$ ← only depends upon v.

$$\frac{dv}{dx} = \frac{f(1,v) - v}{x}$$ separable!

$$\frac{dv}{f(1,v) - v} = \frac{dx}{x}$$

Integrate. Get v if possible.

$$\Rightarrow y = xv.$$

Example: Find the general solution to

Big mess.

$$y' = \frac{x^2 e^{y/x} + y^2}{xy}$$

$y' = f(x,y)$ where $f(x,y) = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Is this 1st order linear? NO (Not yet)

Is it separable? No

Is it Bernoulli? No way.

If it isn't homogeneous, we are sunk!

Check: $f(\alpha x, \alpha y) = \dfrac{(\alpha x)^2 e^{\alpha y/(\alpha x)} + (\alpha y)^2}{(\alpha x)(\alpha y)}$

$$= \frac{\cancel{\alpha^2} x^2 e^{y/x} + \cancel{\alpha^2} y^2}{\cancel{\alpha^2} xy}$$

$$= \frac{x^2 e^{y/x} + y^2}{xy} = f(x,y).$$

$\therefore$ the ODE is homogeneous.

$\therefore$ Substitute $y = xv$

$\Downarrow$

$$y' = \underbrace{\frac{x^2 e^{y/x} + y^2}{xy}}_{= f(x,y)} \implies xv' + v = f(1,v)$$

$$= \frac{e^v + v^2}{v}$$

i.e. $$xv' + v = \frac{1}{v}e^v + v$$

$$xv' = \frac{1}{v}e^v \qquad \text{separable!}$$

$$ve^{-v}\,dv = \frac{1}{x}\,dx$$

Integrate

$$\int ve^{-v}\,dv = \int \frac{1}{x}\,dx$$

$$\boxed{\begin{array}{ll} u = v & du = dv \\ dw = e^{-v}dv & w = -e^{-v} \end{array}}$$

$$-ve^{-v} - \int -e^{-v}\,dv = \ln|x| + C$$

$$-ve^{-v} - e^{-v} = \ln|x| + C$$

Note:

$$\frac{d}{dv}(-ve^{-v}) = -e^{-v} + ve^{-v}$$

$$\frac{d}{dv}(-ve^{-v}) = \frac{d}{dv}e^{-v} + ve^{-v}$$

$$\frac{d}{dv}(-ve^{-v} - e^{-v}) = ve^{-v}$$

Recall $y = xv$

$$v = \frac{1}{x}y$$

$$-\frac{y}{x}e^{-y/x} - e^{-y/x} = \ln|x| + C$$

Gives y implicitly.

You can find more examples worked out in the textbook. ← print and video.

Also, Monday's posted video.

New

Some Applications - Section 2.4

(I will only cover 2 types in this online session. See the text, the video link in the text, and the video posted on the course homepage on Monday for more examples.)

$T(0)=20$

$T(2)=22$

A metal ball at room temperature $20^{o}\,C$ is dropped into a container of boiling water ($100^{o}\,C$). Given that the temperature of the ball increases 2^{o} in the first 2 seconds, find:
(a) The tempurature of the ball after 6 seconds in the boiling water. → Find $T(6)$.
(b) How long does it take for the tempurature of the ball to reach $80^{o}\,C$. Solve $T(t)=80$

Newton's Law of Cooling

(an approximation)

The change in temperature of an object is proportional to the difference in temperature between the environment and the object.

let $T(t)$ be the temperature in degrees C of the metal ball at time t, with $t=0$ corresponding to the time when the ball is placed in the water, and t is measured in seconds.

Newton's law of Cooling:

$$T'(t) = k(100 - T(t))$$

$T(0)=20$

$T(2)=22$

Note: If $w = 100 - T(t)$

$$\Rightarrow w' = -T'(t)$$

$$-w' = kw \Rightarrow \underline{w' = -kw}$$

$$\Rightarrow w = Ce^{-kt}$$

$$\Rightarrow 100 - T(t) = Ce^{-kt}$$

$$\Rightarrow T(t) = 100 - Ce^{-kt}$$

use $T(0) = 20 \Rightarrow 20 = 100 - C \Rightarrow C = 80$

$$\Rightarrow T(t) = 100 - 80e^{-kt}$$

use $T(2) = 22 \Rightarrow 22 = 100 - 80e^{-k\cdot 2}$

$$80e^{-2k} = 78 \Rightarrow e^{-2k} = \frac{78}{80} = \frac{39}{40}$$

$$\Rightarrow e^{-k} = \sqrt{39/40}$$

Note: $e^{-2k} = (e^{-k})^2$

$$\Rightarrow T(t) = 100 - 80\left(\sqrt{39/40}\right)^t$$

i.e. $T(t) = 100 - 80\left(\frac{39}{40}\right)^{t/2}$.

$$\therefore T(6) = 100 - 80\left(\frac{39}{40}\right)^3 = 25.85125$$

Also, $T(t) = 80 \Rightarrow 80 = 100 - 80\left(\frac{39}{40}\right)^{t/2}$

$$\Rightarrow 20 = 80\left(\frac{39}{40}\right)^{t/2} \Rightarrow \left(\frac{39}{40}\right)^{t/2} = \frac{1}{4}$$

$$\ln\left(\left(\frac{39}{40}\right)^{t/2}\right) = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow t = 2\cdot\frac{\ln(1/4)}{\ln(39/40)}$$

$$= 109.5114 \text{ seconds.}$$

$$T(t) = 100 - 80\left(\frac{39}{40}\right)^{t/2}$$

Solve $T(t) = 90$

$$\Rightarrow t = 164.2671074$$

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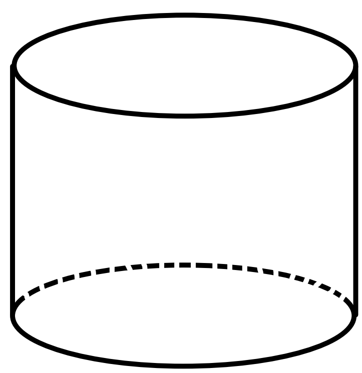
4. How long does it take (from the time the ball is dropped in the water) for the temperature of the ball to reach $90^{\circ}\,C$?

Mixing Problem

See the text

A tank with a capacity of 2m^3 (2000 liters) is initially full of pure water. At time $t = 0$, salt water with salt concentration 5 grams/liter begins to flow into the tank at a rate of 10 liters/minute. The well-mixed solution in the tank is pumped out at the same rate.
(a) Set up, and then solve the initial value problem for the amount of salt in the tank at time t minutes.
(b) Find the time when the salt concentration in the tank becomes 3 grams/liter.



Also, I'll try to post this tomorrow.

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5. 26.21

Approximating Solutions

Direction Fields, Euler's Method, Improved Euler's Method

Section 2.5...make it possible to view the behavior of solutions for many choices of initial data, without writing a formula for the solutions.

Section 2.6...make it possible to approximate the solution to an initial value problem

See video 2 from Monday.

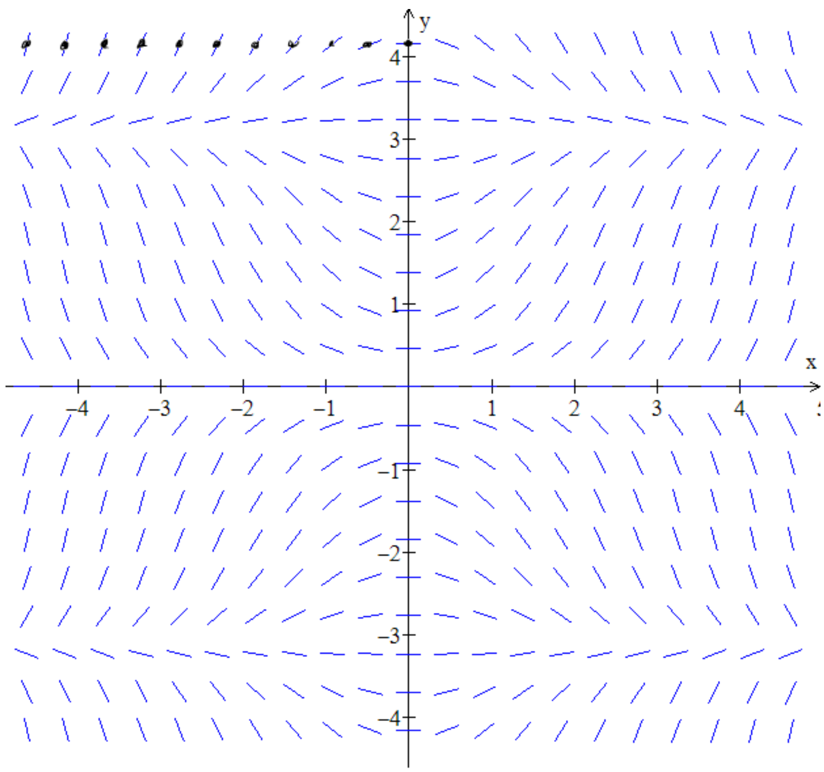
Question: Why do we need something to approximate solutions?

Answer: There are a tremendous number of very important differential equations that have solutions which cannot be written in terms of the simple functions that we have expressions for.

Illustrative Example: The "direction field" is shown below for the differential equation $y' = x \sin(y)$. What do you think this picture represents?

slope $= x \cdot \sin(y)$

slopes at different (x, y) points.



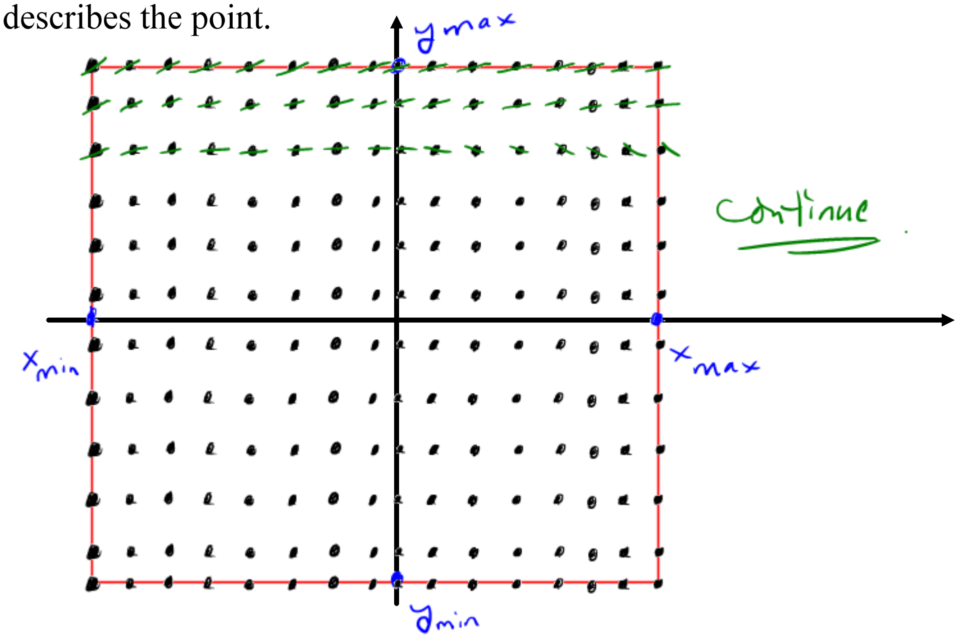
Direction Field Creation Process for $y' = f(x,y)$.

slope $= f(x,y)$

1. Decide where you want to view solutions... Pick a rectangle.

2. Place a rectangular grid of points on this rectangle.

3. Sketch a short line segment at each point that has the slope given by $f(x,y)$, where (x,y) describes the point.



Electronic Options:

1. winplot

2. Find the Polking Java software online

http://math.rice.edu/~dfield/dfpp.html

runs in a browser

Click on $\boxed{\text{DFIELD 2005.10}}$

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6. Give the slope of the line segment in the direction field plot at the point $(-1,2)$ for $y' = x\cos\left(\dfrac{\pi y}{2}\right)$.

plug in $x=-1$, $y=2$

$$\Rightarrow \quad y' = -1 \cdot \cos\left(\frac{\pi \cdot 2}{2}\right) = (-1)(-1) = 1$$

i.e. $\boxed{\text{slope} = 1}$

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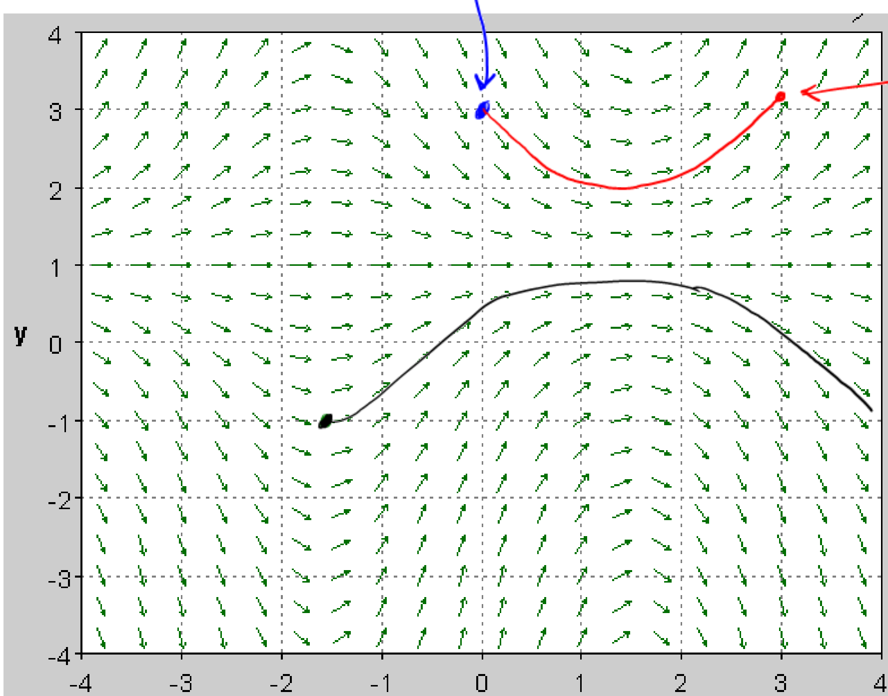
7. The direction field plot is shown for $y' = \cos(x)\,(1 - y)$. Give one of -2, -1, 0, 1, 2, 3, 4 or 5 as a guess for the value of $y\,(3)$, given that $\boxed{y\,(0) = 3.}$

what if $y(-\frac{3}{2}) = -1$



see Monday's video + all of the example videos.

- by hand
- with Excel
- with VBA Excel
- with Matlab

Euler's Method: A crude, simple, and sometimes effective method for approximating the solution to the initial value problem

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Idea: Create an approximate problem by replacing the derivative with a difference quotient.

Method:

1. Select a step size h.
2. Create as many x_i values as necessary.
3. Determine the approximations y_i to $y(x_i)$.

Example: Give the exact solution to the initial value problem

$$y' = x - y$$
$$y(1) = 2$$

Then create the approximation using Euler's method with a step size of $h = .1$ and compare the results to the true solution on the interval [1,2].

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8. Continue the process from the previous example to find an approximation to the solution at $x = 2.1$.

Improved Euler's Method: A more accurate method for approximating the solution to the initial value problem

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

1. Select a step size h.
2. Create as many x_i values as necessary.
3. Determine the approximations y_i to $y(x_i)$.

This will is included in a separate video posted on last Monday morning.

Example: Give the exact solution to the initial value problem

$$y' = x - y$$
$$y(1) = 2$$

Then create the approximation using Improved Euler's method with a step size of $h = .1$, and compare the results to the true solution on the interval [1,2].

This will be included in a separate video posted on Thursday morning...

x_i	z_i	y_i
1		
1.1		
1.2		
1.3		
1.4		
1.5		
1.6		
1.7		
1.8		
1.9		
2		