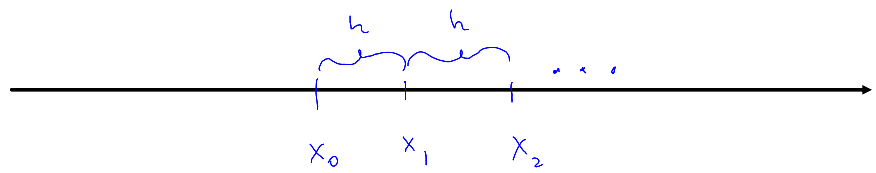
Improved Euler's Method: A more accurate method for approximating the solution to the initial value problem

(IVP) $$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

1. Select a step size h. $\leftarrow$
2. Create as many x_i values as necessary.
3. Determine the approximations y_i to $y(x_i)$.



$x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$, ...

$y_1 \approx y(x_1)$, $y_2 \approx y(x_2)$, $y_3 \approx y(x_3)$, ...

$$y' = f(x,y)$$

$$y(x_0) = y_0$$

$$\int_{x_0}^{x_0+h} y'\,dx = \int_{x_0}^{x_0+h} f(x, y(x))\,dx$$

$$y(x)\Big|_{x_0}^{x_0+h} = \int_{x_0}^{x_0+h} f(x, y(x))\,dx$$

$$y(x_0+h) - y(x_0) = \int_{x_0}^{x_0+h} \underbrace{f(x,y(x))}_{\text{function of } x}\, dx$$

Short interval of integration if h is small.

$$y(x_0+h) - y(x_0) \approx \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_0+h, y(x_0+h))\right]$$

This leads to

$$y_1 - y_0 = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

y_1 appears on both sides

This is <u>implicit</u> ... complicated.

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(Explicit) Improved Euler's:

$$z_1 = y_0 + h\,f(x_0,y_0)$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,z_1)\right]$$

$\vdots$

$$\begin{cases} z_i = y_{i-1} + h\,f(x_{i-1},y_{i-1}) \\ y_i = y_{i-1} + \frac{h}{2}\left[f(x_{i-1},y_{i-1}) + f(x_i,z_i)\right] \end{cases}$$

Example: Give the exact solution to the initial value problem

$$y' = x - y$$
$$\underset{x_0}{y(1)} = \underset{y_0}{2}$$

Then create the approximation using Improved Euler's method with a step size of $h = .1$, and compare the results ~~to the true solution on~~ the interval [1,2].

$$y = x - 1 + 2e^{1-x}$$ (see the previous ex.)

$$f(x,y) = x - y \qquad \begin{cases} z_i = y_{i-1} + h\, f(x_{i-1}, y_{i-1}) \\ y_i = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, z_i)\right] \end{cases}$$

See the video for the computation of z_1 and y_1.

$$z_2 = y_1 + h\, f(x_1, y_1) = 1.91 + .1\,(1.1 - 1.91) = 1.829$$

$$y_2 = y_1 + \frac{.1}{2}\left[f(x_1,y_1) + f(x_2,z_2)\right]$$
$$= 1.91 + \frac{.1}{2}\left[(1.1 - 1.91) + (1.2 - 1.829)\right] = 1.83805$$

x_i	z_i	y_i	
$x_0 =$ 1		$y_0 = 2$	
$x_1 =$ 1.1	$z_1 = 1.9$	$y_1 = 1.91$	
$x_2 =$ 1.2	$z_2 = 1.829$	$y_2 = 1.83805$	
$x_3 =$ 1.3	1.774245	1.78243525	$= y_3$
1.4	1.73419173	1.7416039	$= y_4$
1.5	1.70744351	1.71415153	$= y_5$
1.6	1.69273638	1.69880714	$= y_6$
1.7	1.68892642	1.69442046	$= y_7$
1.8	1.69497841	1.69995051	$= y_8$
1.9	1.70995546	1.71445522	$= y_9$
$x_{10} =$ 2	1.73300969	1.73708197	$= y_{10}$

<u>Note:</u> $y_{10} \approx y(x_{10}) = y(2) = 2 - 1 + 2e^{1-2} = 1 + 2e^{-1}$

$y_{10} = 1.73708197$

$= 1.73575888$

From Excel

xi	zi	yi	exact	error
1		2	2	0
1.1	1.9	1.91	1.90967484	0.00032516
1.2	1.829	1.83805	1.83746151	0.00058849
1.3	1.774245	1.78243525	1.78163644	0.00079881
1.4	1.73419173	1.7416039	1.74064009	0.00096381
1.5	1.70744351	1.71415153	1.71306132	0.00109021
1.6	1.69273638	1.69880714	1.69762327	0.00118386
1.7	1.68892642	1.69442046	1.69317061	0.00124985
1.8	1.69497841	1.69995051	1.69865793	0.00129259
1.9	1.70995546	1.71445522	1.71313932	0.0013159
2	1.73300969	1.73708197	1.73575888	0.00132309

See the video...