Last time we discussed:

- Definitions ✓
- Examples ✓
- Separable equations
- Linear equations

reduce

reduce

Did you try the online solver?

Today:

- Bernoulli and Homogeneous equations
- Applications of linear equations
- Direction fields
- Approximating solutions using Euler's method and Improved Euler's method

Review:

First Order Separable Differential Equations

$$\frac{dy}{dx} = f(x)g(y)$$

1. Separate $\qquad \dfrac{dy}{g(y)} = f(x)\,dx$

2. Integrate $\qquad$ 3. (if possible) solve for y.

First Order Linear Differential Equations

$$\frac{dy}{dx} + p(x)y = f(x)$$

1. Get an integrating factor

$$h(x) = e^{\int p(x)dx}$$ ← any anti-deriv will do

2. multiply both sides by $h(x)$

$$\frac{d}{dx}\left(h(x)y\right) = f(x)\,h(x)$$

3. Integrate

$$h(x)\,y = \int f(x)\,h(x)\,dx$$

4. Solve for y.

Review (continued): **Special Case**

$$\frac{dy}{dx} = \underline{\underline{k}}\, y \quad \Leftrightarrow \quad y = C\,e^{kx}$$

constant

ex. $y' = 2y \Leftrightarrow y = Ce^{2x}$

$u' = -3u \Leftrightarrow u = Ce^{-3x}$

$\frac{dv}{dt} = 4v \Leftrightarrow v = Ce^{4t}$

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1. Give the solution to $y'+3y=0,\ y(0)=4.$

$a.\ 4e^{3t}$

$b.\ 4e^{-3t}$

$c.\ 3e^{4t}$

$d.\ -3e^{4t}$

$e.$ None of these.

$y' = -3y$

$y = ce^{-3t}$

$4 = ce^{-3\cdot 0}$

$c = 4$

$\Rightarrow y = 4e^{-3t}$

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2. Give the value of k below that causes the solution to $y'+ky=2,\ y(0)=4$ to converge to 0.4 as $x\to\infty$.

a. 2

b. 3

c. 4

d. 5

e. None of these.

$$y'+ky=2$$

$$h(x)=e^{\int k\,dx}=e^{kx}$$

$$e^{kx}(y'+ky)=2e^{kx}$$

$$\frac{d}{dx}\left(e^{kx}y\right)=2e^{kx}$$

Integrate $\quad e^{kx}y=\frac{2}{k}e^{kx}+C$

$$y=\frac{2}{k}+Ce^{-kx}$$

$\boxed{\text{Note: } \frac{2}{5}=.4}$

We need $\displaystyle\lim_{x\to\infty}\left(\frac{2}{k}+Ce^{-kx}\right)=\frac{2}{5}$

$\left\{\begin{array}{l} 0 \text{ if } k>0 \\ \cancel{\infty \text{ if } k<0} \end{array}\right.$

$\frac{2}{k}=\frac{2}{5}\Rightarrow k=5>0$

Bernoulli Differential Equations - Section 2.3

(use a substitution to create a first order linear differential equation)

$$\underbrace{y' + p(x)y} = \underbrace{q(x)y^r}, \quad r \neq 0,1$$

Soln idea: Divide by y^r

$$\underbrace{y^{-r}y'}_{\text{nearly } \frac{d}{dx}y^{1-r}} + p(x)\,y^{1-r} = q(x)$$

Note: If I set $\boxed{u = y^{1-r}}$

$$\Rightarrow \frac{du}{dx} = (1-r)y^{-r}\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{1}{1-r}\frac{du}{dx} = y^{-r}\frac{dy}{dx}}$$

$$\Rightarrow \quad \frac{1}{1-r}\frac{du}{dx} + p(x)u = q(x)$$

$$\Rightarrow \frac{du}{dx} + (1-r)p(x)u = q(x)$$

First order linear!

$$y' + p(x)y = q(x)y^r,\ r \neq 0,1$$

Example: Find the general solution to

$$y' - 2y = 3e^{-x}\sqrt{y}$$

Bernoulli!

$p(x) = -2$

$q(x) = 3e^{-x}$

$r = \frac{1}{2}$

$$y^{-\frac{1}{2}}y' - 2y^{\frac{1}{2}} = 3e^{-x}$$

$$u = y^{\frac{1}{2}} \Rightarrow u' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

$$2u' - 2u = 3e^{-x}$$

$$u' - u = \frac{3}{2}e^{-x}$$

$$h(x) = e^{\int -1\,dx} = e^{-x}$$

$$e^{-x}(u' - u) = \frac{3}{2}e^{-x} \cdot e^{-x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}u\right) = \frac{3}{2}e^{-2x}$$

Integrate: $e^{-x}u = -\frac{3}{4}e^{-2x} + C$

$$\Rightarrow u = -\frac{3}{4}e^{-x} + Ce^{x}$$

Recall: $u = y^{\frac{1}{2}}$

$$\Rightarrow y^{\frac{1}{2}} = -\frac{3}{4}e^{-x} + Ce^{x}$$

$$\Rightarrow y = \left(-\frac{3}{4}e^{-x} + Ce^{x}\right)^2$$

Homogeneous Differential Equations - Section 2.3

(use a substitution to create a first order separable differential equation)

special

$$y' = f(x,y) \leftarrow 1^{st} \text{ order ODE}$$

special $\left\{ \begin{array}{l} \text{where } f(\alpha x, \alpha y) = f(x,y) \\ \text{for all } \alpha, x, y. \end{array} \right.$

Idea : Create a new function v via $y = xv$.

$$y' = xv' + v$$

Substitute :

$$xv' + v = f(x, xv)$$

$$= f(x \cdot 1, x \cdot v)$$

$$xv' + v = f(1, v)$$

homog. property

$$xv' = f(1,v) - v$$

$$v' = \frac{f(1,v) - v}{x}$$

Separable! Solve it.

Example: Find the general solution to

$\rightarrow f(x,y) = \dfrac{-2y}{x+\sqrt{xy}}$

Not separable

Not linear

Not Bernoulli

$$\frac{dy}{dx} = \frac{-2y}{x+\sqrt{xy}}$$

Is it homogeneous? yes

check $f(\alpha x, \alpha y) = \dfrac{-2\alpha y}{\alpha x + \sqrt{\alpha^2 xy}}$

$= \dfrac{-2\alpha y}{\alpha x + \alpha\sqrt{xy}}$ if $\alpha > 0$

$= \dfrac{-2y}{x+\sqrt{xy}} = f(x,y)$

Subst: $y = xv \Rightarrow y' = xv' + v$

Subst. $xv' + v = \dfrac{-2xv}{x+\sqrt{x^2 v}}$

We'll solve for $x > 0$.

$= \dfrac{-2xv}{x + x\sqrt{v}} = \dfrac{-2v}{1+\sqrt{v}}$

Note: $\sqrt{x^2} = \begin{cases} x, & \text{if } x>0 \\ -x, & \text{if } x<0 \end{cases}$

$$xv' + v = \frac{-2v}{1+\sqrt{v}}$$

$$xv' = \frac{-2v}{1+\sqrt{v}} - v$$

$$xv' = \frac{-2v - v - v^{3/2}}{1+\sqrt{v}}$$

$$\Rightarrow \frac{1+\sqrt{v}}{-2v - v - v^{3/2}} v' = \frac{1}{x}$$

Separable! Find v.

Then use $y = xv$ to get y.

You can find more examples worked out in the textbook.

Some Applications - Section 2.4

(I will only cover 2 types in the online session. See the text and the video link in the text for others.)

Last Night - We did one Newton's law of cooling problem, and one mixing problem.

Find the orthogonal trajectories for the family of curves $x^2 + y^2 = 2Cx$

Notes: 1. we get a different curve for each value of C.

2. The family of orthogonal trajectories consist of curves whose intersection with curves in the given family is always at a right angle. i.e. tangent lines at intersections are perpendicular.

i.e. slopes are negative reciprocals of each other.

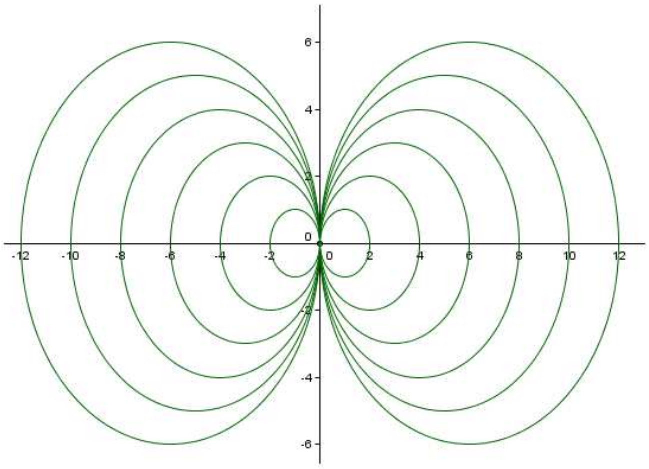
Q: What do the curves in $x^2+y^2 = 2Cx$ look like?

arbitrary

A: $x^2 - 2Cx + C^2 + y^2 = 0 + C^2$

$$(x-C)^2 + y^2 = C^2$$

Circles centered at $(C,0)$ with radius $|C|$.



Find $\frac{dy}{dx}$ for $\boxed{x^2 + y^2 = 2\underline{\underline{C}}\, x}$

Implicit diff. $2x + 2y\frac{dy}{dx} = 2C$

$$\cancel{2}y\frac{dy}{dx} = \cancel{2}C - \cancel{2}x$$

At this point, you might be tempted to set the dy/dx for the new family of curves (the so called orthogonal trajectories) equal to the negative reciprocal of this value.
WAIT!!! Get rid of the C first!!

$$\frac{dy}{dx} = \frac{C-x}{y}$$

$$C = \frac{x^2+y^2}{2x}$$

$$\frac{dy}{dx} = \frac{\frac{x^2+y^2}{2x} - x}{y}$$

$$= \frac{x^2+y^2-2x^2}{2xy}$$

$\frac{dy}{dx}$ for our original family, ind. of C.

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$

For the new family: We need

orthogonal trajectories

$$\frac{dy}{dx} = \frac{-2xy}{\underbrace{y^2-x^2}_{\text{negative reciprocal of above.}}}$$

Solve $\quad \frac{dy}{dx} = \frac{-2xy}{y^2-x^2}$

Solve $\frac{dy(x)}{dx} = -\frac{2\,x\,y(x)}{-x^2+y(x)^2}$: homogeneous

Let $y(x) = x\,v(x)$, which gives $\frac{dy(x)}{dx} = x\,\frac{dv(x)}{dx} + v(x)$:

$$x\,\frac{dv(x)}{dx} + v(x) = -\frac{2\,x^2\,v(x)}{-x^2+x^2\,v(x)^2}$$

Simplify:

$$x\,\frac{dv(x)}{dx} + v(x) = -\frac{2\,v(x)}{v(x)^2-1}$$

Solve for $\frac{dv(x)}{dx}$:

$$\frac{dv(x)}{dx} = -\frac{(v(x)^2+1)\,v(x)}{x\,(v(x)^2-1)}$$

Simplify:

$$\frac{dv(x)}{dx} = -\frac{v(x)^3+v(x)}{x\,(v(x)^2-1)}$$

Divide both sides by $\frac{v(x)^3+v(x)}{v(x)^2-1}$:

$$\frac{\frac{dv(x)}{dx}\,(v(x)^2-1)}{v(x)^3+v(x)} = -\frac{1}{x}$$

Integrate both sides with respect to x:

$$\int \frac{\frac{dv(x)}{dx}\,(v(x)^2-1)}{v(x)^3+v(x)}\,dx = \int -\frac{1}{x}\,dx$$

Evaluate the integrals: $\log \equiv \ln$

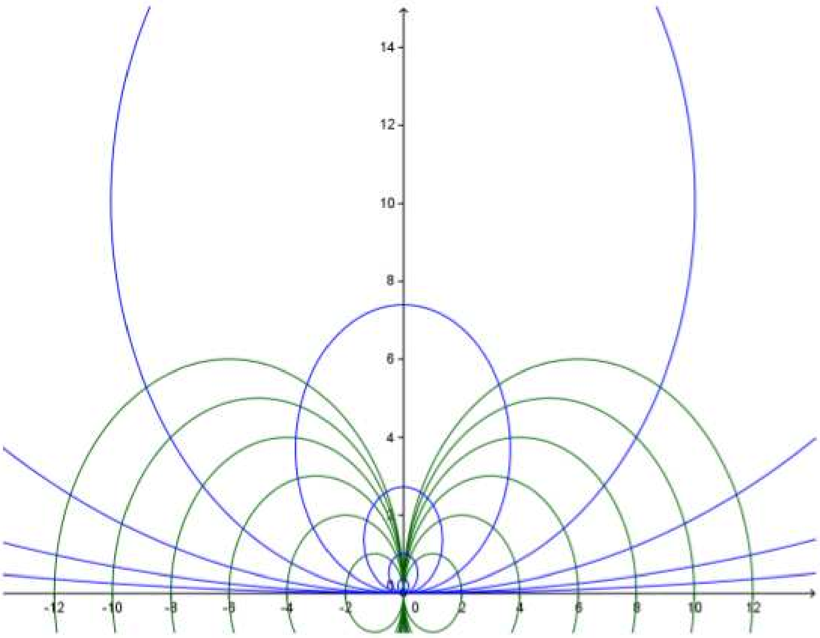
$\log(v(x)^2+1) - \log(v(x)) = -\log(x) + c_1$, where c_1 is an arbitrary constant.

Solve for $v(x)$:

Subs $y(x) = x\,v(x)$:

$$y(x) = \frac{1}{2}\left(e^{c_1} - \sqrt{e^{2c_1} - 4x^2}\right) \text{ or } y(x) = \frac{1}{2}\left(e^{c_1} + \sqrt{e^{2c_1} - 4x^2}\right)$$

$\pm \qquad \pm$



A flu virus is spreading through a city with a population of 25,000. The disease is spreading at a rate proportional to the product of the number of people who have it and the number who do not. Suppose that 100 people had the flu initially and that 400 people had it after 10 days.

$\hookrightarrow P(10) = 400$ $\hookrightarrow P(0) = 100$

(a) How many people will have the flu after 20 days?
(b) How long will it take for half the population to have the flu?

$P(t) \equiv$ # people with the flu at time t in days.

$P'(t) \equiv$ rate of change of $P(t)$

$$P'(t) = k\,P(t)\,(25000 - P(t))$$

separable ODE

This is a mathematical model for the sentence above.

$$P(0) = 100, \quad P(10) = 400$$

a) Find $P(20)$. We need $P(t)$.

$$\frac{dP}{P(25000-P)} = k\,dt$$

integrate.

$$\underbrace{\int \frac{dP}{P(25000-P)}}\; = kt + C$$

$$\frac{1}{P(25000-P)} = \frac{A}{P} + \frac{B}{25000-P}$$

Just for now, treat P like a variable.

$$1 = A(25000-P) + BP$$

Killer P values: $\boxed{P = 25000}$, $P = 0$

Subst $P = 25000$

$1 = 25000\,B \Rightarrow B = +\frac{1}{25000}$

$\boxed{P = 0}$

Subst $1 = 25000\,A \Rightarrow A = \frac{1}{25000}$

$\therefore$

$$\frac{1}{P(25000-P)} = \frac{1/25000}{P} + \frac{1/25000}{25000-P}$$

$$\Rightarrow \int \frac{dP}{P(25000-P)} = kt + C$$

becomes

$$\int \left(\frac{1/25000}{P} + \frac{1/25000}{25000-P} \right) dP = kt + C$$

$$\frac{1}{25000} \ln(P) - \frac{1}{25000} \ln(25000-P) = kt + C$$

$P > 0$ $\qquad$ $P < 25000$

$$\frac{1}{25000} \ln\left(\frac{P}{25000-P}\right) = kt + C.$$

use $P(0) = 100$ $\qquad$ $P(10) = 400$

$$\frac{1}{25000} \ln\left(\frac{100}{24900}\right) = C$$

$$\frac{1}{25000} \ln\left(\frac{P}{25000-P}\right) = kt + \frac{1}{25000} \ln\left(\frac{100}{24900}\right)$$

$$\frac{1}{25000} \ln\left(\frac{400}{24600}\right) = 10k + \frac{1}{25000} \ln\left(\frac{100}{24900}\right)$$

$$\Rightarrow \frac{1}{25000}\left[\ln\left(\frac{2}{123}\right) - \ln\left(\frac{1}{249}\right)\right] = 10k$$

$$\frac{1}{250000}\left[\ln\left(\frac{498}{123}\right)\right] = k$$

Answer a) Find P when $t = 20$.

$$\frac{1}{25000} \ln\left(\frac{P}{25000 - P}\right) = kt + \frac{1}{25000} \ln\left(\frac{100}{24900}\right)$$

$$\frac{1}{250000}\left[\ln\left(\frac{498}{123}\right)\right] = k$$

Put this in for k, 20 in for t and solve for P.

Approximating Solutions

geometric

Direction Fields, **Euler's Method, Improved Euler's Method**

Section 2.5...make it possible to view the behavior of solutions for many choices of initial data, without writing a formula for the solutions.

Section 2.6...make it possible to approximate the solution to an initial value problem

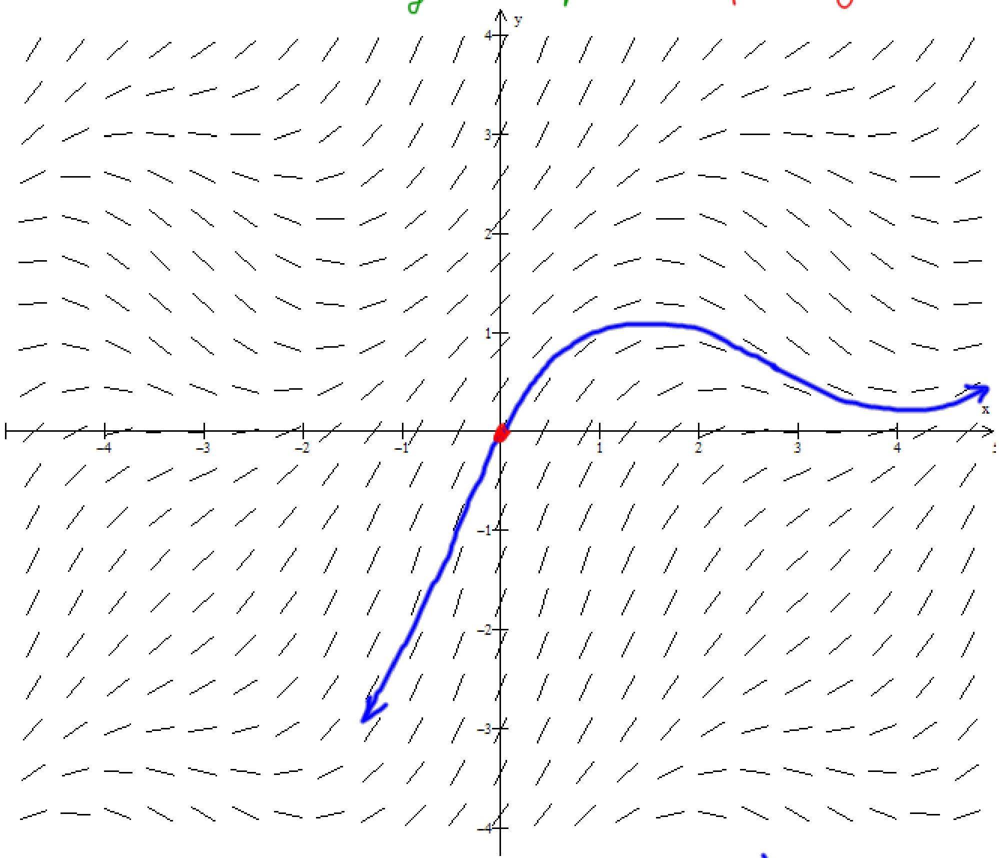
See Wed. video + extra videos.

Question: Why do we need something to approximate solutions?

Answer: There are a tremendous number of very important differential equations that have solutions which cannot be written in terms of the simple functions that we have expressions for.

Illustrative Example: The "direction field" is shown below for the differential equation $y' = -\sin(y) + \cos(x) + 1$. What do you think this picture represents?

$y' \equiv$ slope.   Spse $y(0) = 0$



The line segments are tangents.
The curve is an approximation to the sol'n of $y' = -\sin(y) + \cos(x) + 1$, $y(0) = 0$

Direction Field Creation Process for $y' = f(x,y)$.

slope at (x,y) is $f(x,y)$

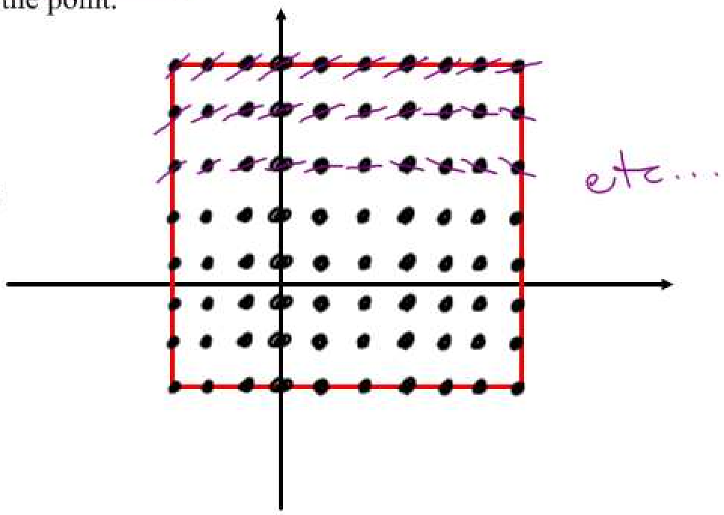
1. Decide where you want to view solutions... Pick a rectangle.

2. Place a rectangular grid of points on this rectangle.

3. Sketch a short line segment at each point that has the slope given by f(x,y), where (x,y) describes the point.

tangents

These short line segments (tangents) are used to guide a solution to an initial value problem.



Electronic Options:

1. winplot

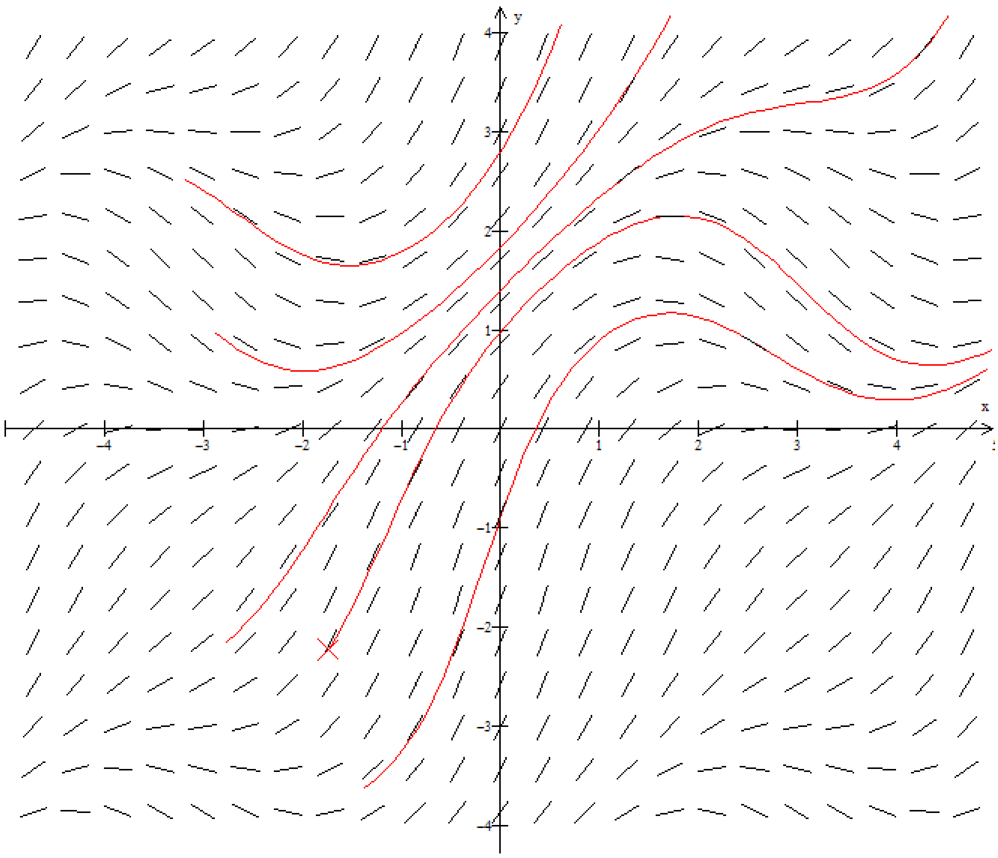
2. Find the Polking Java software online

http://math.rice.edu/~dfield/dfpp.html

Click on $\boxed{\text{DFIELD 2005.10}}$

$y' = x\sin(y)$, $y(0) = 1$ sol'n curve

winplot.... Google it.



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3. Give the slope of the line segment in the direction field plot at the point $(1,-2)$ for the differential equation $y' = x^2 + y$.

a. -2

b. 2

c. -1

d. 1

e. None of these.

slope at $(1,-2)$

i.e. when $x=1$, $y=-2$

$y' = 1^2 + (-2) = -1$

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4. The direction field is shown below for the differential equation $y' = x - \sin(y)$. Sketch the solution given by the initial data $y(-2) = 0$, and then approximate $y(2)$.

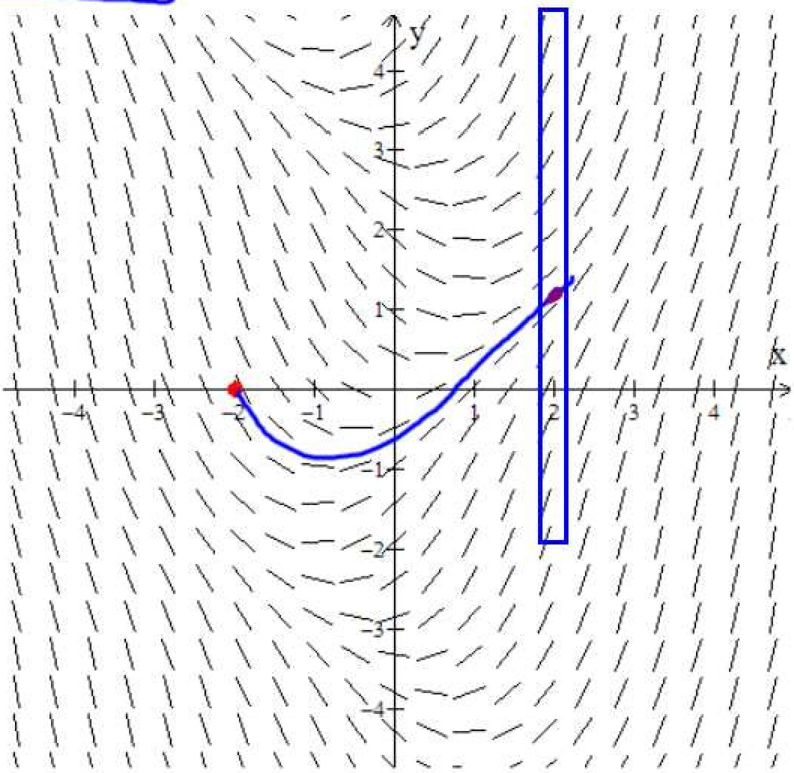
a. -2

b. 3

c. 0

d. 1

e. None of these.



Euler's Method: A crude, simple, and sometimes effective method for approximating the solution to the initial value problem

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

See the videos

Implementation:

1. Select a step size h.
2. Create as many x_i values as necessary.
3. Determine the approximations y_i to $y(x_i)$.

Example: Create an approximation to the solution to the intial value problem on the interval [0,1] by using Euler's method with a step size of $h = .1$.

$$y' = x - y^2$$
$$y(0) = 1$$

you

Try it.

Improved Euler's Method: A more accurate method for approximating the solution to the initial value problem

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

5. B
6. C
7. B
8. A

1. Select a step size h.
2. Create as many x_i values as necessary.
3. Determine the approximations y_i to $y(x_i)$.

Posted in a video on Wednesday night.

Example: Give the exact solution to the initial value problem

$$y' = x - y$$
$$y(1) = 2$$

Then create the approximation using Improved Euler's method with a step size of $h = .1$, and compare the results to the true solution on the interval [1,2].

Posted in a video on Wednesday night.

x_i	z_i	y_i
1		
1.1		
1.2		
1.3		
1.4		
1.5		
1.6		
1.7		
1.8		
1.9		
2		