Info

- You should have completed Online Quizzes 1, 2 and 3.
- We are starting chapter 3 today.
- I posted several videos associated with solving problems from chapter 3.
- Assignment 3 is posted.
- The Alternate Assignment for this week will be posted on Friday.

Important: You must read and watch videos outside of the live class meetings. It is impossible to learn all of the material in 2 hours each week.

Open EMCF03

$y = Ce^{-3x} \Rightarrow y = 2e^{-3x}$

EMCF03

1. Give the solution to $y' = -3y$, $y(0) = 2$.
   Then give the value of $y(1)$.

Accurate to 4 decimal places.

$y(1) = 2e^{-3} = \underline{0.0995}$

Chapter 3

Linear Second-Order Differential Equations

$$y'' + p(x)y' + q(x)y = f(x)$$

$p(x)$, $q(x)$, $f(x)$ all known.

$y \equiv$ unknown

Related Terminology: $\boxed{\text{Coefficients}}$, $\boxed{\text{forcing term}}$, homogeneous (reduced) equation, nonhomogeneous equation.

↳ $f \equiv 0$

↳ f is nonzero

We will see that the homogeneous differential equation is the key to solving the nonhomogeneous differential equation.

Note: "Homogeneous" in this setting has NOTHING to do with the term we learned about studying first order ODEs.

$L[y]$

Question: Why is

$$\underbrace{y''+p(x)y'+q(x)y}_{L[y]}=f(x)$$

called a linear second order differential equation?

y'' is the highest derivative.

?? Define $L[y] = y'' + p(x)y' + q(x)y$.

For example: If $p(x) = \cos(x)$ and $q(x) = x-1$.

ex. Then $L[y] = y'' + \cos(x)\,y' + (x-1)\,y$

So, $L[\underbrace{x^2+2x-4}_{y}] = 2 + \cos(x)(2x+2) + (x-1)(x^2+2x-4)$

$y' = 2x+2$
$y'' = 2$

Properties of L ? General Case.

Recall $L[y] = y'' + p(x)y' + q(x)y$

Spse u and v are functions.

$$L[\underbrace{u+v}_{y}] = (u+v)'' + p(x)(u+v)' + g(x)(u+v)$$

$$= u'' + v'' + p(x)(u'+v') + g(x)u + g(x)v$$

$$= u'' + p(x)u' + g(x)u + v'' + p(x)v' + g(x)v$$

$$= L[u] + L[v]$$

Also, if α is a scalar, then
$\hookrightarrow$ a number

$$L[\alpha u] = (\alpha u)'' + p(x)(\alpha u)' + g(x)(\alpha u)$$

$$= \alpha u'' + p(x)\alpha u' + g(x)\alpha u$$

$$= \alpha\left(u'' + p(x)u' + g(x)u\right)$$

$$= \alpha L[u]$$

These 2 properties are "linear" properties.

Hence "linear" in the naming of the ODE.

$$y'' + p(x)y' + q(x)y = f(x)$$ ← 2^{nd} order linear ODE

$$L[y] = y'' + p(x)y' + q(x)y$$ ← linear differential operator

Properties: If u and v are twice differentiable functions and α is a scalar, then

(i): $L[u+v] = L[u] + L[v]$

(ii): $L[\alpha u] = \alpha L[u]$

These properties make it possible to solve these differential equations.

Remark: The homogeneous and nonhomogeneous equations are linked in an important way.

Nonhomog ODE $\longrightarrow$

$$y''+p(x)y'+q(x)y=f(x) \qquad \text{(NH)}$$

Associated Homogeneous counterpart $\longrightarrow$

$$y''+p(x)y'+q(x)y=0 \qquad \text{(H)}$$

$\longleftrightarrow$ Associated "reduced" equation.

Namely, **if you can solve the homogeneous equation, then you can solve the nonhomogeneous equation.**

((**Fact:** (NH) and (H) have infinitely many solutions.

ex. $y''+xy'+\cos(x)y=e^x-1$

has infinitely many sol'ns.

ex. $y''+2y'-3y=e^{-x}$

has infinitely many sol'ns.

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2. Give the number of solutions to $y'' - xy' + 2y = \sin(x)$.

If the answer is 1, then input 1.
If the answer is 2, then input 2. etc...
If the answer is infinitely many, then input 999.

infinitely many.

i.e. 999

$p(x), q(x), f(x)$ known.

x_0, b, m known

Second Order Linear Initial Value Problem

$$\boxed{y''+p(x)y'+q(x)y} = f(x)$$

$$y(x_0) = b \qquad L[y]$$

$$y'(x_0) = m$$

*

Fact: A second order linear initial value problem has exactly one solution.

Why?

Idea

Spse you have 2 sol'ns, u and v.

$$\boxed{\begin{aligned} L[u] &= f(x) \\ u(x_0) &= b \\ u'(x_0) &= m \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} L[v] &= f(x) \\ v(x_0) &= b \\ v'(x_0) &= m \end{aligned}}$$

Then

Fact: The initial value problem above has a unique solution (i.e. exactly one solution).

So, $L[u-v] = L[u] - L[v] = f(x) - f(x) = 0$

i.e. $u-v$ solves $L[u-v] = 0$.

$(u-v)(x_0) = u(x_0) - v(x_0) = b-b = 0$

$(u-v)'(x_0) = u'(x_0) - v'(x_0) = m-m = 0$.

i.e. $u-v$ solves

$$\left[\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= 0 \\ y'(x_0) &= 0 \end{aligned}\right. \rightarrow \underline{\underline{u = v}}.$$

It is not hard to show y must be 0. i.e. $u-v=0$

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3. Give the number of solutions to

$$y'' - xy' + 2y = \sin(x), y(0) = 2, y'(0) = -1.$$

If the answer is 1, then input 1.
If the answer is 2, then input 2. etc...
If the answer is infinitely many, then input 999.

1

General Solutions to Linear Second Order Differential Equations

$p(x)$, $q(x)$, $f(x)$ known.

y unknown

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{(NH)}$$

The general solution to the nonhomogeneous differential equation is

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{general sol'n of the reduced DE}} + \underbrace{z}_{\text{any particular sol'n of the nonhomog. DE}}.$$

i.e. general sol'n of $y'' + p(x)y' + q(x)y = 0$

Just any function that solves (NH)

Term: Particular solution.

Question: Why does the solution break up this way?

Spse $y = w + z$ where z solves (NH).

We know $L[y] = f(x)$

$$L[w + z] = f(x)$$

$$L[w] + L[z] = f(x)$$

i.e. $L[w] + f(x) = f(x)$

$$\Leftrightarrow L[w] = 0.$$

$\boxed{\text{General Solutions}}$ and Fundamental Sets of Solutions to Homogeneous Second-Order Differential Equations

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

The set $\{y_1, y_2\}$ is a fundamental set of solutions to (H) provided

y_1 and y_2 solve (H)

and

y_1 and y_2 are $\boxed{\text{Linearly independent}}$.

Neither is a multiple of the other.

The general solution to (H) is given by $y = c_1 y_1 + c_2 y_2$ provided

arbitrary constant

$\{y_1, y_2\}$ is a fundamental set of sol'ns to (H).

Terms: Linear independence, Wronskian, fundamental matrix.

Fact: 2 functions are linearly independent on an interval if and only if the Wronskian is nonzero on the interval.

Note: $\{\sin(x), \cos(x)\}$ is a Linearly independent set because $\sin(x)$ is not a constant multiple of $\cos(x)$ and visa versa.

$y'' + y = 0$.

$\sin(x)$ solves

$\cos(x)$ solves

0 solves

$\{\sin(x), \cos(x)\}$ is a fundamental set of sol'ns.

$\{0, \cos(x)\}$ is NOT a L.I. set.

b/c $0 = 0 \cdot \cos(x)$

i.e. 0 is a constant multiple of $\cos(x)$.

So, the general sol'n to $y'' + y = 0$ is $y = C_1 \sin(x) + C_2 \cos(x)$.

Spse y_1 and y_2 solve $L[y] = 0$.
The fundamental matrix assoc. with y_1 and y_2 is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Note: The determinant of this matrix is $\boxed{y_1 y_2' - y_1' y_2}$

Wronskian of y_1 and y_2

Spse y_1 and y_2 solve $L[y] = 0$.
The fundamental matrix assoc. with y_2 and y_1 is

$$\begin{pmatrix} y_2 & y_1 \\ y_2' & y_1' \end{pmatrix}.$$

Wronskian of y_2 and y_1 is $y_2 y_1' - y_2' y_1$

y_1 and y_2 are L.I. iff their Wronskian is nonzero.

constant coef. a, b known real numbers.

Special Case: $y'' + a\,y' + b\,y = 0.$

Illustrative Example:

Give the general solution to $\boxed{y'' + 4y = 0.}$

$\sin(2x)$ and $\cos(2x)$ solve.

Q: Is $\{\sin(2x), \cos(2x)\}$ a fundamental set of sol'ns to $y'' + 4y = 0$?

A: sol'ns ✓ L.I. ... hm .... Yes. ⇒ It is a FS of sol'ns.

Note: $\det\begin{pmatrix} \sin(2x) & \cos(2x) \\ 2\cos(2x) & -2\sin(2x) \end{pmatrix} = -2\sin^2(2x) - 2\cos^2(2x) = -2.$

Comment: The Wronskian of two solutions is either always zero or never zero.

∴ the general sol'n to $y'' + 4y = 0$ is $y = C_1\sin(2x) + C_2\cos(2x)$ where C_1 and C_2 are arbitrary constants.

Give the solution to $y''+4y=0,\ y(0)=1,\ y'(0)=-1.$

we know $y = C_1 \sin(2x) + C_2 \cos(2x)$ is

the general sol'n to $y''+4y=0$.

Let's take care of the initial data.

$\underline{y(0)=1}$: $C_1 \sin(2\cdot 0) + C_2 \cos(2\cdot 0) = 1$

$\underline{\text{i.e.}}$ $C_2 = 1$.

$\underline{y'(0)=-1}$: $y' = 2C_1 \cos(2x) - 2C_2 \sin(2x)$

$2C_1 \cos(2\cdot 0) - 2C_2 \sin(2\cdot 0) = -1$

$\Rightarrow C_1 = -\frac{1}{2}$.

$\therefore$ our sol'n is

$$\boxed{y = -\frac{1}{2}\sin(2x) + \cos(2x)}$$

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4. Give the solution to $y''+4y=0,\ y(0)=1,\ y'(0)=-1$ at $x=\pi$.

$y = -\frac{1}{2}\sin(2x) + \cos(2x)$

$\Rightarrow \quad y(\pi) = 1.$

Special Case: $y'' + a\,y' + b\,y = 0.$

Another Illustrative Example:

Give the general solution to $y'' - y = 0.$

e^{-x} and e^{x} solve.

Q: Is $\{e^{-x}, e^{x}\}$ a fundamental set of sol'ns?

A: Solutions? ✓

L.I.?

$$\det\begin{pmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{pmatrix} = e^{-x}e^{x} - -e^{-x}e^{x}$$

$$= 1 + 1$$

$$= 2 \neq 0.$$

$\boxed{e^{-x}e^{x} = e^{-x+x} = e^{0} = 1}$

yes. we get L.I.

$\therefore$ the general sol'n to $y'' - y = 0$ is

$$y = c_1 e^{-x} + c_2 e^{x}.$$

Give the solution to $y''-y=0,\ y(0)=1,\ y'(0)=-1.$

The general sol'n to $y''-y=0$ is

$$y = c_1 e^{-x} + c_2 e^{x}.$$

$y(0)=1$: $c_1 e^{0} + c_2 e^{0} = 1$

$\Rightarrow \boxed{c_1 + c_2 = 1}$

$y'(0)=-1$: $y' = -c_1 e^{-x} + c_2 e^{x}$

$-c_1 e^{0} + c_2 e^{0} = -1$

$\boxed{-c_1 + c_2 = -1}$

2 equations:

$$\boxed{\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + c_2 &= -1 \end{aligned}} \Rightarrow$$

Add

$2c_2 = 0$

$\Rightarrow \boxed{c_2 = 0.}$

$c_1 + c_2 = 1$

$\Rightarrow \boxed{c_1 = 1}$

$\therefore\ y = e^{-x}$ is the unique solution.

Special Case: $y'' + a\,y' + b\,y = 0.$

Solution Process:

The truth $\leftrightarrow$ 1 Start with the polynomial equation

$$\underbrace{r^2 + ar + b}_{} = 0.$$

Characteristic polynomial associated with $y'' + ay' + by = 0$

associated characteristic equation.

2 Get the roots.

Cases: (i) real roots

distinct real roots r_1 and r_2.

general sol'n $= c_1 e^{r_1 x} + c_2 e^{r_2 x}$

repeated real root r_1

general sol'n $= c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

(ii) Complex roots.

↳ next page.

$\alpha + \beta i$ and $\alpha - \beta i$

$\beta \neq 0$

general sol'n $= c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

ex. $y'' + 2y' - 3y = 0.$

Linear, second order, constant coefficient, homogeneous ODE.

Solve $r^2 + 2r - 3 = 0.$

$(r+3)(r-1) = 0$

$r = -3, \quad r = 1.$ ← distinct real roots

$\therefore$ the general sol'n is

$$y = c_1 e^{-3x} + c_2 e^{x}$$

ex. $y'' + y' + y = 0.$

Solve $r^2 + r + 1 = 0$

$r = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

complex roots.

$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}.$

General sol'n

$$y = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

ex. $y'' + 2y' + y = 0.$

Solve $r^2 + 2r + 1 = 0.$

$(r+1)^2 = 0$

$r = -1$ ← repeated

General Sol'n: $y = c_1 e^{-x} + c_2 x e^{-x}.$

Show that $\left\{e^{-2x}, e^{-3x}\right\}$ is a fundamental set of solutions to $y''+5y'+6y=0$.

Sol'ns? You do it.

L.I.? $\det\begin{pmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{pmatrix} = -3e^{-5x}+2e^{-5x}$

$= -e^{-5x}$

never zero.

$\therefore$ L.I.

$\therefore \left\{e^{-2x}, e^{-3x}\right\}$ is a F.S. of Sol'ns.

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5. The solution to $y''+5y'+6y=0, y(0)=1, y'(0)=-1$

has the form $y=c_1e^{-2x}+c_2e^{-3x}$. Give the value of c_1.

$y(0)=1$: $\boxed{c_1+c_2=1}$

$y'(0)=-1$: $y'=-2c_1e^{-2x}-3c_2e^{-3x}$

$\boxed{-2c_1-3c_2=-1}$

$3\cdot(\text{first})+(\text{second})$

$\boxed{c_1=2}$

Give the general solution to $y''+2y'-15y=0.$

you do it.

Give the general solution to $y'' - 4y' + 4y = 0.$

Solve $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$ ← repeated.

$y = c_1 e^{2x} + c_2 x e^{2x}$

Back to the general case...

The Wronskian

Definition: (recall)

the Wronskian of y_1 and y_2

$$\underbrace{W[y_1, y_2]} = y_1 y_2' - y_2 y_1' = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

Note: $W[y_2, y_1] = y_2 y_1' - y_1 y_2' = -W[y_1, y_2]$

Special case - when y_1 and y_2 solve

$y_2'' + p(x) y_2' + q(x) y_2 = 0$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

$$W[y_1, y_2] = C\, e^{-\int p(x)dx}$$

why? $W[y_1, y_2] = \underline{y_1 y_2'} - \underline{y_2 y_1'}$

$$(W[y_1, y_2])' = y_1 y_2'' + \underbrace{y_1' y_2' - (y_2 y_1'' + y_2' y_1')}_{\text{Cancel.}}$$

$$= y_1 y_2'' - y_2 y_1''$$

$$= \underbrace{y_1(-p(x)y_2' - q(x)y_2) - y_2(-p(x)y_1' - q(x)y_1)}_{\text{Cancel}}$$

$$= y_1(-p(x)\,y_2') - y_2(-p(x)y_1')$$

$$= -p(x)\,(y_1 y_2' - y_2 y_1') = -p(x) W[y_1, y_2].$$

i.e. $$(W[y_1, y_2])' = -p(x)\, W[y_1, y_2]$$

$$(\underbrace{W[y_1, y_2]}_{u})' + p(x)\, \underbrace{W[y_1, y_2]}_{u} = 0.$$

$$u' + p(x)\,u = 0$$

Integrating factor! $$\mu(x) = e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(e^{\int p(x)dx}\, u\right) = 0$$

Integrate $$e^{\int p(x)dx}\, u = C_1$$

$$\Rightarrow \quad u = C_1 e^{-\int p(x)dx}$$

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6. $W\left[e^{2x}, e^{-x}\right] =$ $\det\begin{pmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{pmatrix} = -e^{x} - 2e^{x}$

$= -3e^{x}$

(0) $-3e^x$
(1) $2e^x$
(2) $-e^x$
(3) e^x
(4) $-4e^x$
(5) None of these.

Summarizing the Wronskian Information

Theorem

Suppose that y_1 and y_2 are solutions of $y'' + py' + qy = 0$, and let $W = W[y_1, y_2]$. Then:

i. $W' = -pW$

ii. $W = Ce^{-\int p(x)dx}$

iii. $W(x) \neq 0$ for all $x \in \mathcal{I}$ when y_1, y_2 are linearly independent;

$W(x) = 0$ for all $x \in \mathcal{I}$ when y_1, y_2 are $\underbrace{\text{linearly dependent}}_{\text{Not L.I.}}$.

Give the general form of the Wronskian of any pair of solutions to $y''+\frac{2}{x}y'-4y=0$.

If y_1 and y_2 solve, then

$$W[y_1, y_2] = C_1 e^{-\int \frac{2}{x}dx}$$

$$= C_1 e^{-2\ln|x|} = C_1 e^{\ln\left(\frac{1}{x^2}\right)}$$

$$= \frac{C_1}{x^2}.$$

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7. Give the general form of the Wronskian for any pair of solutions to $y''+2y'-3\cos(x)y=0$.

$$W[y_1, y_2] = C e^{-\int 2\,dx}$$

$$= C e^{-2x}$$

(0) Ce^{2x}

(1) Ce^{-2x}

(2) $Ce^{-3\sin(x)}$

(3) $Ce^{3\sin(x)}$

(4) None of these.

It can be very difficult to find a fundamental set of solutions to

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

when p and q are not constants.

However, sometimes one nontrivial solution can be found. When this is the case, a fundamental set can be obtained by using a process called

Reduction of Order

Reduction of Order - Used to find a second linearly independent solution to a homogeneous linear second-order differential equation from a given nontrivial solution.

Idea: Suppose y_1 is a nontrivial solution to

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

and we want to find another nontrivial solution so that $\{ y_1 , y_2 \}$ is a fundmental set of solutions.

Process: We know $W[y_1, y_2] = C_1 e^{-\int p(x)\,dx}$

Let's make a strategic choice of C_1.

$$C_1 = 1.$$

$$\underbrace{y_1}_{\text{known}} y_2' - \underbrace{y_1'}_{\text{known}} y_2 = \underbrace{e^{-\int p(x)\,dx}}_{\text{known}}$$

1st order linear ODE for y_2. Solve it!

we know it from earlier is

Show that e^{2x} is a solution to $y''-4y'+4y=0$.
Then use reduction of order to find a second linearly independent solution.

(**Note:** Two more examples of reduction of order are given in a posted video.)

Solve $W[e^{2x}, y_2] = e^{-\int -4\,dx} = e^{4x}$

$$e^{2x} y_2' - 2e^{2x} y_2 = e^{4x}$$

$$y_2' - 2y_2 = e^{2x}$$

integrating factor : $\mu(x) = e^{\int -2\,dx} = e^{-2x}$

$$\frac{d}{dx}\left(e^{-2x} y_2\right) = e^{-2x} e^{2x} = 1$$

Integrate $\Rightarrow$ $e^{-2x} y_2 = x + C_1$

$\Rightarrow y_2 = xe^{2x} + Ce^{2x}$

choose any C we want. $C=0$.

$\Rightarrow \underline{y_2 = xe^{2x}}$.

$\therefore \{e^{2x}, xe^{2x}\}$ is a F.S. of sol'ns.

We already knew this. I just wanted to show you the method in action.

Solving the Nonhomogeneous Problem
The Method of Variation of Parameters

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{(NH)}$$

Idea:

1. Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to

$$y'' + p(x)y' + q(x)y = 0$$

2. Then a particular solution to

$$y'' + p(x)y' + q(x)y = f(x)$$

can be written in the form

$$z = u(x)y_1 + v(x)y_2$$

The method of variation of parameters gives a mechanism for finding a particular solution from 2 linearly independent solutions y_1 and y_2 to the associated homogeneous equation (i.e. the reduced equation).

i.e. the homogeneous problem RULES!!

Theorem Let y_1 and y_2 be linearly independent solutions of

$$y'' + py' + qy = 0,$$

and let $W = W[y_1, y_2] = y_1 y_2' - y_1' y_2$. If u and v satisfy

$$u' = -\frac{y_2}{W} f \quad \text{and} \quad v' = \frac{y_1}{W} f,$$

then

$$z = u\,y_1 + v\,y_2$$

is a solution of

$$y'' + py' + qy = f.$$

(i.e. z is a particular solution)

$\therefore$ the general sol'n to (NH) is

$$y = C_1 y_1 + C_2 y_2 + z.$$

Give the general solution to $y''+y=\csc(x)$. (NH)

Process:

1. Get a fundamental set of solutions to $y''+y=0$.

Solve $r^2+1=0 \Rightarrow r=\pm i$ $\{\cos(x), \sin(x)\}$ is a F.S. of sol'ns

2. Get a particular solution of the form $z = u\,y_1 + v\,y_2$

where $u' = -\dfrac{y_2}{W} f$ and $v' = \dfrac{y_1}{W} f$.

$W[y_1,y_2] = \det\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} = \cos^2(x)+\sin^2(x) = 1$

$u' = \dfrac{-\sin(x)}{1}\csc(x) = -1. \Rightarrow u = -x$

$v' = \dfrac{\cos(x)}{1}\csc(x) = \cot(x)$ $\quad v = \ln|\sin(x)|$

3. Write the general solution of (NH).

$\therefore\ z = -x\cos(x) + (\ln|\sin(x)|)\sin(x)$

$\Rightarrow$ the general sol'n sol'n to $y''+y=\csc(x)$ is

$y = c_1\cos(x) + c_2\sin(x) + \left(-x\cos(x) + \sin(x)\ln|\sin(x)|\right).$

Remark: There is a process that can be used to solve the nonhomogeneous problem in the SPECIAL CASE

$$y'' + ay' + by = f(x)$$

where a and b are real numbers AND $f(x)$ is made up of sums and products of

$\exp(kx)$, $\sin(mx)$, $\cos(nx)$, x, 1

The method is called the method of undertermined coefficients. (see the text)

Guessing Method.

I posted some videos.

Read the Online Text, Watch the Embedded Videos, and Look at the Posted Videos