EMCF03b

1. Give the solution to $y' = 2y,\ y(0) = -3.$

$a.\ 2e^{-3t}$

$b.\ -2e^{-3t}$

$c.\ -3e^{2t}$

$d.\ 3e^{-2t}$

$e.$ None of these.

$y = Ce^{2t} \longrightarrow -3 = Ce^{2\cdot 0}$

$= C$

$C = -3$

$\Rightarrow \boxed{y = -3e^{2t}}$

Linear Second-Order Differential Equations

coefficients

$$y'' + p(x)y' + q(x)y = f(x)$$

forcing term

y is unknown.

$p(x)$, $q(x)$, $f(x)$ are known.

Related Terminology: Coefficients, forcing term, homogeneous (reduced) equation, nonhomogeneous equation.

nonhomogeneous: $f(x) \not\equiv 0$.

homogeneous: $f(x) \equiv 0$

* **We will see** that the homogeneous differential equation is the key to solving the nonhomogeneous differential equation. *

Ex: 1. $y'' + xy' - e^{-2x}y = 0$ is homogeneous

2. $y'' + xy' - e^{-2x}y = \cos(x)$ is nonhomogeneous.

Question: Why is

$$\boxed{y'' + p(x)y' + q(x)y} = f(x)$$

called a linear second order differential equation?

Duh

Obvious

b/c it is a 2^{nd} order ODE

??

See next page.

$$y'' + p(x)y' + q(x)y = f(x)$$

ODE

Differential operator

$$\underline{\underline{L}}[y] = y'' + p(x)y' + q(x)y$$

function that takes in functions, and spits out functions.

Properties: If u and v are twice differentiable functions and α is a scalar, then

(i): $L[u+v] = L[u] + L[v]$

(ii): $L[\alpha u] = \alpha L[u]$

These properties make it possible to solve these differential equations.

L is a linear transformation.

Why?

$$L[u+v] = (u+v)'' + p(x)(u+v)' + q(x)(u+v)$$

$$= u'' + v'' + p(x)u' + p(x)v' + q(x)u + q(x)v$$

$$= L[u] + L[v]$$

You should verify (ii).

Ex. Spse $L[y] = y'' + \sin(x)y' - 3y$

Then $L[e^{-x}] = e^{-x} + \sin(x)(-e^{-x}) - 3e^{-x}$

$\frac{d}{dx}e^{-x} = -e^{-x}$ $\qquad = e^{-x}\left(1 - \sin(x) - 3\right)$

$\frac{d^2}{dx^2}e^{-x} = e^{-x}$ $\qquad = e^{-x}\left(-\sin(x) - 2\right).$

Remark: The homogeneous and nonhomogeneous equations are linked in an important way.

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{(NH)}$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

Namely, **if you can solve the homogeneous equation, then you can solve the nonhomogeneous equation.**

Fact: (NH) and (H) have infinitely many solutions.

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2. There is only one solution to $y'' - 2e^{-x}y' + \frac{3}{x}y = \cos(x)$.

a. True

b. False

No.

There are infinitely many.

Second Order Linear Initial Value Problem

$$y'' + p(x)y' + q(x)y = f(x)$$

initial data

$$\begin{cases} y(x_0) = b \\ y'(x_0) = m \end{cases}$$

Fact: A second order linear initial value problem has exactly one solution.

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3. There are at least two solutions to

$$y''-2e^{-x}y'+\frac{3}{x}y=\cos(x), y(1)=2,\ y'(1)=0.$$

a. True

b. False

NO.

There is exactly one sol'n.

General Solutions to Linear Second Order Differential Equations

$$\overbrace{y''+p(x)y'+q(x)y}^{L[y]}=f(x) \qquad \text{(NH)}$$

The general solution to the nonhomogeneous differential equation is

$$y = c_1y_1 + c_2y_2 + z.$$

general soln of the reduced DE → $c_1y_1 + c_2y_2$

any particular soln of the nonhomog. DE → z

Homogeneous $\quad y''+p(x)y'+q(x)y=0. \quad$ (H)

Term: Particular solution.

Question: Why does the solution break up this way?

b/c If u and v both solve (NH) then

$$\begin{aligned} L[u-v] &= L[u+(-1)v] \\ &= L[u]+L[(-1)v] \\ &= L[u]-L[v] \\ &= f(x)-f(x) \\ &= 0 \end{aligned}$$

i.e. $u-v$ solves (H).

$\therefore$ $u = (\text{sol'n to (H)}) + v$.

General Solutions and Fundamental Sets of Solutions to Homogeneous Second-Order Differential Equations

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

The set $\{y_1, y_2\}$ is a fundamental set of solutions to (H) provided

y_1 and y_2 both solve (H)

and

y_1 and y_2 are not constant multiples of each other.

The general solution to (H) is given by $y = c_1y_1 + c_2y_2$ provided

$\{y_1, y_2\}$ is a fundamental set of sol'ns to H.

Terms: Linear independence, Wronskian, fundamental matrix.

Fact: 2 functions are linearly independent on an interval if and only if the Wronskian is nonzero on the interval.

ex. $\cos(x)$ and $\sin(x)$ are linearly independent

b/c $\cos(x) \neq k\sin(x)$

and $\sin(x) \neq \tilde{k}\cos(x)$.

ex. $\cos(x)$ and 0 are linearly dependent

b/c $0 = 0 \cdot \cos(x)$.

i.e. The zero function is a constant multiple of $\cos(x)$.

Fundamental Matrix associated with 2 solutions y_1 and y_2 of

$$y'' + p(x)y' + q(x)y = 0$$

is $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$. $\leftarrow$ 2x2.

$$\det\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = 2\cdot3 - (-1)\cdot1 = 7$$

If y_1 and y_2 are any 2 functions Then the Wronskian of y_1 and y_2 is

$$W[y_1, y_2] = y_1y_2' - y_1'y_2.$$

Note: $W[y_1, y_2] \neq W[y_2, y_1]$.

$$W[y_1, y_2] = y_1y_2' - y_1'y_2 \qquad W[y_2, y_1] = y_2y_1' - y_2'y_1$$

These are negatives of each other.

i.e. $W[y_1, y_2] = -W[y_2, y_1]$

Note: If $\{y_1, y_2\}$ is a fundamental set of sol'ns to

$$y'' + p(x)y' + q(x)y = 0 \quad (H)$$

then the general sol'n to (H) is

$$c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are arbitrary constants.

Ex. $W[e^x, \sin(x)]$

$$= e^x \cos(x) - e^x \sin(x)$$

$$W[e^{2x}, e^{-3x}] = e^{2x} \cdot (-3e^{-3x}) - (2e^{2x})e^{-3x}$$

$$= -5e^{2x}e^{-3x}$$

$$= -5e^{-x}$$

Special Case: $y'' + \underline{a}\, y' + \underline{b}\, y = 0.$ constant coef. 2nd order linear homogeneous diff. eq.

real #s

Illustrative Example:

Give the general solution to $\underline{y'' - 4y = 0}$.

special case $a = 0 \quad b = -4$

1. Need 2 L.I. sol'ns.

are these L.I.? e^{2x} is a sol'n. e^{-2x} is a sol'n

yes. $W[e^{2x}, e^{-2x}] = e^{2x}(-2e^{-2x}) - 2e^{2x}e^{-2x}$

$= -4 \neq 0$

$\Rightarrow e^{2x}$ and e^{-2x} are L.I.

$\therefore \{e^{2x}, e^{-2x}\}$ is a fundamental set of sol'ns.

2. $\therefore$ the general sol'n to

$y'' - 4y = 0$

is $c_1 e^{2x} + c_2 e^{-2x}$

where c_1 and c_2 are arbitrary constants.

Give the solution to $\underbrace{\boxed{y''-4y=0}\ \boxed{y(0)=1,\ y'(0)=2.}}_{\text{Initial value Problem}}$

note: The general sol'n to $y''-4y=0$

is $y=\underline{c_1}e^{2x}+\underline{c_2}e^{-2x}$.

Use the initial data to "pin down"

c_1 and c_2.

$\underline{y(0)=1}$: $1=c_1e^0+c_2e^0$

i.e. $\boxed{1=c_1+c_2}$

$\underline{y'(0)=2}$: $\longrightarrow$ $2=2c_1e^0-2c_2e^0$

$y'=2c_1e^{2x}-2c_2e^{-2x}$ i.e. $\boxed{1=c_1-c_2}$

Subtract : $0=2c_2 \Rightarrow c_2=0$

Add : $2=2c_1 \Rightarrow c_1=1$

$\therefore \boxed{y=e^{2x}}$

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4. Give the solution to $y''-4y=0,\ y(0)=1,\ y'(0)=2$ at $x=1$.

$y = e^{2x}$

a. e^{-2}

b. e^{2}

c. e

d. e^{-1}

e. None of these.

Special Case: $y'' + a\,y' + b\,y = 0.$

Another Illustrative Example:

Give the solution to $y'' + y = 0$

i.e. give the general sol'n.

We need 2 L.I. sol'ns.

Note: $\cos(x)$ solves

$\sin(x)$ solves.

Q: Are these L.I.?

A: $W[\cos(x), \sin(x)] = \cos(x)\cdot\cos(x) - (-\sin(x))\sin(x)$

$= \cos^2(x) + \sin^2(x)$

$= 1 \neq 0$

$\therefore$ $\cos(x)$, $\sin(x)$ are L.I.

$\Rightarrow$ $\{\cos(x), \sin(x)\}$ is a fundamental set of solutions to $y'' + y = 0$.

$\therefore$ The general sol'n is

$$\boxed{c_1\cos(x) + c_2\sin(x).}$$

Give the solution to $\underbrace{y''+y=0,\ y(0)=1,\ y'(0)=2}_{\text{IVP}}$.

The general sol'n to $y''+y=0$ is

$$y = c_1\cos(x) + c_2\sin(x)$$

now use the initial data.

$y(0)=1$: $\quad 1 = c_1\cos(0) + c_2\sin(0) = c_1$

i.e. $c_1 = 1$.

$y'(0)=2$: $\quad 2 = -c_1\sin(0) + c_2\cos(0)$

$y' = -c_1\sin(x) + c_2\cos(x)$ $\quad = c_2$

i.e. $c_2 = 2$

$$\Rightarrow \boxed{y = \cos(x) + 2\sin(x).}$$

Special Case: $y'' + a y' + b y = 0.$ a and b are real #s

Solution Process: In general.

1. Find the roots of $\underline{\underline{r^2 + ar + b}}$.

the characterist polynomial

2. $\underline{\underline{3 \text{ cases}}}$

(i) 2 distinct real roots λ_1 and λ_2

Then $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$ is a fundamental set. So the general sol'n is

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

(ii) 1 repeated real root λ_1.

i.e. $r^2 + ar + b = (r - \lambda_1)^2$

Then $\{e^{\lambda_1 x}, x e^{\lambda_1 x}\}$ is a fundamental set. So the general sol'n is

$$c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

(iii) We have complex roots

$\mu + i\omega$ and $\mu - i\omega$

where μ and ω are real numbers and $\omega \neq 0$. Then

$\{ e^{\mu x}\cos(\omega x), e^{\mu x}\sin(\omega x) \}$ is a fundamental set. So the general sol'n is

$$c_1 e^{\mu x}\cos(\omega x) + c_2 e^{\mu x}\sin(\omega x).$$

Give a fundamental set of solutions to $y''+3y'-4y=0$, and then give the general solution to this differential equation.

Characteristic polynomial:

$$r^2+3r-4$$

Roots: Solve $r^2+3r-4=0$

$$(r+4)(r-1)=0$$

$$r=-4,\quad r=1 \leftarrow \text{2 distinct real roots}$$

$\therefore \{e^{-4x}, e^{x}\}$ is a fundamental set.

Also, the general sol'n is

$$c_1e^{-4x}+c_2e^{x}.$$

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5. The solution to $y''+5y'+6y=0, y(0)=1, y'(0)=-1$ has the form $y=c_1e^{-2x}+c_2e^{-3x}$. Give the value of c_1.

a. -2
b. -1
c. 0
d. 1
e. 2
f. None of these.

$r^2+5r+6=0$

$(r+2)(r+3)=0 \quad r=-2, r=-3$

$\Rightarrow y=c_1e^{-2x}+c_2e^{-3x}$ is the general sol'n to $y''+5y'+6y=0$.

$y(0)=1$

$1=c_1+c_2$

$y'(0)=-1 \quad y'=-2c_1e^{-2x}-3c_2e^{-3x}$

$-1=-2c_1-3c_2$

3(First eq.)+(second eq.)

$\boxed{2=c_1}$

Give the general solution to $y''-2y'+4y=0$.

Solve $r^2-2r+4=0$

$$r=\frac{2\pm\sqrt{4-16}}{2}$$

$$=\frac{2\pm\sqrt{-12}}{2}=1\pm\frac{2\sqrt{-3}}{2}$$

$$=1\pm\sqrt{-3}=1\pm\sqrt{3}\,i$$

$\boxed{i=\sqrt{-1}}$

i.e. $1+\sqrt{3}\,i$ and $1-\sqrt{3}\,i$

Complex roots!

$\therefore\ \{e^x\cos(\sqrt{3}x),\ e^x\sin(\sqrt{3}x)\}$ is a fundamental set.

$\Rightarrow$ the general sol'n is

$$\boxed{c_1e^x\cos(\sqrt{3}x)+c_2e^x\sin(\sqrt{3}x).}$$

Give the general solution to $y''-2y'+y=0$.

Solve $r^2 - 2r + 1 = 0$.

$(r-1)^2 = 0$

$\Rightarrow$ $r=1$ is a repeated real root.

$\therefore$ $\{e^x, xe^x\}$ is a fundamental set.

$\Rightarrow$ the general sol'n is

$c_1 e^x + c_2 x e^x$.

Back to the general case... $\longrightarrow y'' + p(x)y' + q(x)y = 0.$

The Wronskian

Definition: (recall)

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

Special case - when y_1 and y_2 solve

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

$$W[y_1, y_2] = C_1 e^{-\int p(x)\,dx}$$

ALWAYS!! Amazing!!

Why? $(W[y_1, y_2])' = (y_1 y_2' - y_1' y_2)'$

$$= (y_1 y_2')' - (y_1' y_2)'$$

$$= y_1 y_2'' + y_1' y_2' - (y_1' y_2' + y_1'' y_2)$$

$$= y_1 y_2'' - y_1'' y_2$$

Recall y_1 and y_2 solve

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

$$y_1'' = -p(x)y_1' - q(x)y_1$$

$$y_2'' = -p(x)y_2' - q(x)y_2$$

$$= y_1(-p(x)y_2' - q(x)y_2) - (-p(x)y_1' - q(x)y_1)y_2$$

$$(W[y_1, y_2])' = -p(x)(y_1 y_2' - y_1' y_2)$$

$$= -p(x)\, W[y_1, y_2]$$

$$\Rightarrow W[y_1, y_2] = C_1 e^{-\int p(x)\,dx}$$

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6. $W\left[e^{-x}, e^{2x}\right] =$ $e^{-x} \cdot 2e^{2x} - (-e^{-x})e^{2x}$

a. $3e^{x}$

b. $-3e^{x}$

c. $2e^{-x}$

d. $-2e^{-x}$

e. None of these.

$= 3e^{-x}e^{2x}$

$= 3e^{x}$.

Summarizing the Wronskian Information

Theorem

Suppose that y_1 and y_2 are solutions of $y'' + py' + qy = 0$, and let $W = W[y_1, y_2]$. Then:

i. $W' = -pW$

ii. $W = Ce^{-\int p(x)dx}$

iii. $W(x) \neq 0$ for all $x \in \mathcal{I}$ when y_1, y_2 are linearly independent;
$W(x) = 0$ for all $x \in \mathcal{I}$ when y_1, y_2 are linearly dependent.

Give the general form of the Wronskian of any pair of solutions to $y''+xy'-\frac{3}{x}y=0$.

$p(x)=x$

$$W[y_1,y_2]=Ce^{-\int x\,dx}$$

$$=Ce^{-\frac{x^2}{2}}.$$

EMCF03b

7. Give the general form of the Wronskian for any pair of solutions to $y''+2y'-3\cos(x)y=0$.

a. Ce^{2x}

b. Ce^{-2x}

c. $Ce^{-3\sin(x)}$

d. $Ce^{3\sin(x)}$

e. None of these.

$p(x)=2$

$W[y_1,y_2]=Ce^{-\int 2\,dx}$

$=Ce^{-2x}$

It can be very difficult to find a fundamental set of solutions to

$$y''+p(x)y'+q(x)y=0 \qquad \text{(H)}$$

when p and q are not constants.

However, sometimes one nontrivial solution can be found. When this is the case, a fundamental set can be obtained by using a process called

i.e. y is not the zero function

Reduction of Order

How? Spse y_1 is a nontrivial sol'n to $y''+p(x)y'+q(x)y=0$

we can find a second L.I. sol'n as follows:

Solve $W[y_1, y_2] = C_1 e^{-\int p(x)dx}$ for our favorite choice of C_1.

i.e. $C_1 = 1$.

i.e. we solve

$$y_1 y_2' - y_1' y_2 = e^{-\int p(x)dx}$$

Known

$\Rightarrow$ Solve this first order ode to get y_2.

Reduction of Order - Used to find a second linearly independent solution to a homogeneous linear second-order differential equation from a given nontrivial solution.

Idea: Suppose y_1 is a nontrivial solution to

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

and we want to find another nontrivial solution so that $\{ y_1 , y_2 \}$ is a fundmental set of solutions.

Process: See last page.

The differential equation $y''+2y'+y=0$ has linearly independent solutions e^{-x} and xe^{-x}. Pretend you only know that e^{-x} is a solution, and use reduction of order to find a second linearly independent solution.

y_1

(**Note:** Two more examples of reduction of order are given in a posted video.)

$$p(x) = 2$$

Solve $W[e^{-x}, y_2] = e^{-\int 2\,dx}$

$$e^{-x}y_2' - (-e^{-x})y_2 = e^{-2x}$$

$$y_2' + y_2 = e^{-x}$$

First order linear ode. Get any nontrivial sol'n.

$$h(x) = e^{x}$$

$$\frac{d}{dx}(e^{x}y_2) = e^{x}e^{-x} = 1$$

Integrate $e^{x}y_2 = x$

$$\Rightarrow y_2 = xe^{-x}$$

$\therefore \{e^{-x}, xe^{-x}\}$ is a fundamental set.

8. B
9. B
10. B

Solving the Nonhomogeneous Problem
The Method of Variation of Parameters

$$y''+p(x)y'+q(x)y=f(x) \qquad \text{(NH)}$$

General sol'n is

Idea:

$$y = c_1 y_1 + c_2 y_2 + z$$

where $\{y_1, y_2\}$ is a fundamental set of solutions to

$$y''+p(x)y'+q(x)y=0$$

and z is a particular sol'n to (NH).

Then a particular solution to

$$y''+p(x)y'+q(x)y=f(x)$$

can be written in the form

$$z=u(x)y_1+v(x)y_2$$

The method of variation of parameters gives a mechanism for finding a particular solution from 2 linearly independent solutions y_1 and y_2 to the associated homogeneous equation (i.e. the reduced equation).

i.e. the homogeneous problem RULES!!

Theorem Let y_1 and y_2 be linearly independent solutions of

$$y'' + py' + qy = 0,$$

and let $W = W[y_1, y_2] = y_1 y_2' - y_1' y_2$. If u and v satisfy

$$u' = -\frac{y_2}{W} f \quad \text{and} \quad v' = \frac{y_1}{W} f,$$

then

$$z = u\, y_1 + v\, y_2$$

is a solution of

$$y'' + py' + qy = f.$$

(i.e. z is a particular solution)

Give the general solution to $y''+y=\tan(x)$. (NH) $f(x)$

Process: $\{\cos(x), \sin(x)\} \leftarrow r^2+1=0$, $r=\pm i$

1. Get a fundamental set of solutions to $y''+y=0$.

$\{\cos(x), \sin(x)\}$ y_1 y_2

2. Get a particular solution of the form $z = u\,y_1 + v\,y_2$

where $u' = -\frac{y_2}{W} f$ and $v' = \frac{y_1}{W} f$.

$W[\cos(x), \sin(x)] = \cdots = 1$

3. Write the general solution of (NH).

$$u' = \frac{-\sin(x)}{1} \cdot \tan(x) = \frac{-\sin^2(x)}{\cos(x)}$$

$$v' = \frac{\cos(x)}{1} \tan(x)$$

$$v' = \sin(x)$$

$$v = -\cos(x)$$

$$u = \int \frac{-\sin^2(x)}{\cos(x)}\,dx$$

$$= \int \frac{\cos^2(x)-1}{\cos(x)}\,dx$$

$$= \int (\cos(x) - \sec(x))\,dx$$

$$u = \sin(x) - \ln|\sec(x)+\tan(x)|$$

$u = \sin(x) - \ln|\sec(x) + \tan(x)|$

$v = -\cos(x)$

$z = u\,y_1 + v\,y_2$

Recall: $y_1 = \cos(x)$

$y_2 = \sin(x)$

$\Rightarrow$ a particular sol'n is

$$z = \left(\sin(x) - \ln|\sec(x) + \tan(x)|\right)\cos(x) + (-\cos(x))\sin(x)$$

$$= -\cos(x)\ln|\sec(x) + \tan(x)|$$

$\therefore$ the general sol'n is

$$\boxed{y = c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|}$$

Remark: There is a process that can be used to solve the nonhomogeneous problem in the SPECIAL CASE

y'' + ay' + by = f(x)

where a and b are real numbers AND f(x) is made up of sums and products of

exp(kx), sin(mx), cos(nx), x, 1

The method is called the method of undertermined coefficients.
(see the text)

Also known as

GUESSING.

Read the Online Text, Watch the Embedded Videos, and Look at the Posted Videos

3.1 – 3.5