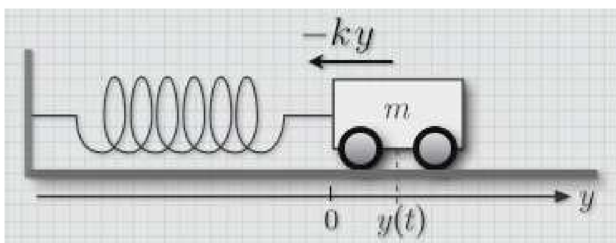
Section 3.6

Simple Applications of Second Order Linear Differemtial Equations

Spring Mass Systems

Part I: No damping and no external forces - Simple Harmonic Motion.



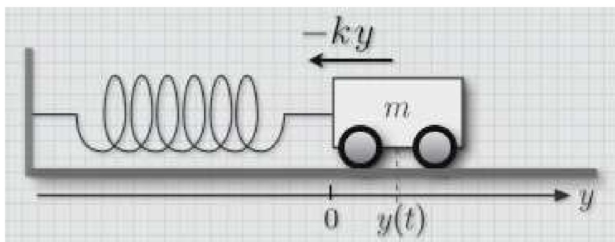
Hooke's law states that for small displacements, the restoring force is proportional to the displacement.

$$m\,y'' + k\,y = 0.$$

$$\omega = \sqrt{k/m}$$

Terms: Period, frequency, amplitude, phase shift.

Part II: Damping, but no external forces.



$$m\,y'' = -k\,y - \delta\,y'$$

$$m\,y'' + \delta\,y' + k\,y = 0$$

Terms: Overdamped, underdamped, critically damped.

The Forced System

$$m\,y'' + \delta\,y' + k\,y = F(t)$$

Special Case: $F(t) = F_0 \cos(\gamma t)$

Terms: Natural frequency = $\omega/(2\pi)$, Applied frequency = $\gamma/(2\pi)$.

Higher Order Linear Differential Equations

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

$$L[y(x)] = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x)$$

Terms: Nonhomogeneous equation, homogeneous equation, linear differential operator, number of solutions for the homogeneous and nonhomogeneous equations.

Initial Value Problem

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x);$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ \ldots,\ y^{(n-1)}(a) = \alpha_{n-1}$$

Uniqueness Theorem:

Finding the General Solution to the Homogeneous Equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

Terms: Linear combination of solutions, linear independence, Wronskian, fundamental set of solutions.

Finding the General Solution to the *Constant Coefficient* Homogeneous Equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

Examples: Find the general solution of the differential equation.

$$y^{(6)} - y'' = 0$$

Examples: Find the solution of the initial value problem.

$$y''' - y'' + 9y' - 9y = 0; \quad y(0) = y'(0) = 0,\ y''(0) = 2$$

Examples: Find a homogeneous differential equation of least order that has the following function as a solution.

$$y = 2e^{2x} + 3\sin x - x$$

Finding the General Solution to the Nonhomogeneous Equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + z(x)$$

Term: Particular solution.

Finding a Particular Solution in the Constant Coefficient Case

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Term: Undetermined coefficients, variation of parameters.

Example: Find the general solution to

$$y^{(4)} - y = 2e^x + \cos x$$

Laplace Transforms

Motivation: Laplace transforms can be used to turn linear constant coefficient differential equations into algebraic equations.

DEFINITION Let f be a continuous function on $[0, \infty)$. The Laplace transform of f, denoted by $\mathcal{L}[(f(x)]$, or by $F(s)$, is the function given by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx. \tag{1}$$

The domain of F is the set of all real numbers s for which the improper integral converges.

Illustrative Examples:

$$L\left[f(x)\right]=\int_0^\infty e^{-sx}f(x)\,dx$$

$$L\left[e^{x}\right]=$$

$$L\left[e^{2x}\right]=$$

$$L\left[e^{ax}\right]=$$

Recall: $$L\left[f(x)\right]=\int_0^\infty e^{-sx}f(x)\,dx$$

As a result...

1. $L\left[\alpha f(x)\right]=$

2. $L\left[f(x)+g(x)\right]=$

$$L[\, y'(x)\,] =$$

$$L[\, y''(x)\,] =$$

Example:

Find the Laplace transform of the solution to

$$y''+3y'-4y=3e^{-x}+2,\ y(0)=2,\ y'(0)=-1.$$

We will do this directly, without finding the solution first!!

Recall:

$$L\left[y'(x)\right]=-y(0)+sL\left[y(x)\right]$$

$$L\left[y''(x)\right]=-y'(0)-s\,y(0)+s^2L\left[y(x)\right]$$

More Examples:

$$L[1] =$$

$$L[x] =$$

$$L[\cos(x)] =$$

$$L[\sin(x)] =$$

Table of Laplace Transforms

Add others, or create your own!!

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\dfrac{1}{s}, \qquad s > 0$
$e^{\alpha x}$	$\dfrac{1}{s-\alpha}, \qquad s > \alpha$
$\cos \beta x$	$\dfrac{s}{s^2+\beta^2}, \qquad s > 0$
$\sin \beta x$	$\dfrac{\beta}{s^2+\beta^2}, \qquad s > 0$
$e^{\alpha x} \cos \beta x$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \qquad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-r)^{n+1}}, \qquad s > r$
$x \cos \beta x$	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}, \qquad s > 0$
$x \sin \beta x$	$\dfrac{2\beta s}{(s^2+\beta^2)^2}, \qquad s > 0$

$$L\left[y'(x)\right] = -y(0) + sL\left[y(x)\right]$$

$$L\left[y''(x)\right] = -y'(0) - s\,y(0) + s^2 L\left[y(x)\right]$$

$$L\left[f(x)\right] = \int_0^\infty e^{-sx} f(x)\,dx$$

$$L\left[2\sin(x)-3e^{-x}+1\right]=$$

$$L\left[3e^{2x}\cos(3x)+x\sin(x)-3x^2\right]=$$

Note: $L[f(x)\,g(x)]$ is generally NOT equal to $L[f(x)]\,L[g(x)]$.

Example: Use the Laplace transform to solve

$$y''-y'-6y=e^{-x},\ y(0)=2,\ y'(0)=3.$$

(Note: We can do this easier, without Laplace transforms, but I want to illustrate the process.)

Recall:

$$L\left[y'(x)\right]=-y(0)+sL\left[y(x)\right]$$

$$L\left[y''(x)\right]=-y'(0)-s\,y(0)+s^2L\left[y(x)\right]$$

The TRUTH!!

The Laplace Transform is typically used to solve problems of the form

$$\begin{cases} y''+ay'+by=f(t) \\ y(0)=b,\ y'(0)=m \end{cases}$$

where $f(t)$ is a piecewise defined function.