Section 3.6

Simple Applications of Second Order Linear Differential Equations

Spring Mass Systems

Part I: No damping and no external forces - Simple Harmonic Motion.

Hooke's law states that for small displacements, the restoring force is proportional to the displacement.

\[ my'' + ky = 0. \]

\[ \omega = \sqrt{k/m} \]

Terms: Period, frequency, amplitude, phase shift.
**Part II:** Damping, but no external forces.

\[ m \dddot{y} = -k \dot{y} - \delta y' \]

\[ m \dddot{y} + \delta \dot{y} + k y = 0 \]

**Terms:** Overdamped, underdamped, critically damped.
The Forced System

\[ m \ddot{y} + \delta \dot{y} + k y = F(t) \]

**Special Case:** \( F(t) = F_0 \cos(\gamma t) \)

**Terms:** Natural frequency = \( \omega/(2\pi) \), Applied frequency = \( \gamma/(2\pi) \).
Higher Order Linear Differential Equations

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]

\[ L[y(x)] = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) \]

**Terms:** Nonhomogeneous equation, homogeneous equation, linear differential operator, number of solutions for the homogeneous and nonhomogeneous equations.
Initial Value Problem

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x); \]

\[ y(a) = \alpha_0, \; y'(a) = \alpha_1, \; \ldots, \; y^{(n-1)}(a) = \alpha_{n-1} \]

Uniqueness Theorem:
Finding the General Solution to the Homogeneous Equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]

Terms: Linear combination of solutions, linear independence, Wronskian, fundamental set of solutions.
Finding the General Solution to the *Constant Coefficient* Homogeneous Equation

\[ y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0 y = 0 \]
Examples: Find the general solution of the differential equation.

\[ y^{(6)} - y'' = 0 \]
Examples: Find the solution of the initial value problem.

\[ y''' - y'' + 9y' - 9y = 0; \quad y(0) = y'(0) = 0, \ y''(0) = 2 \]
Examples: Find a homogeneous differential equation of least order that has the following function as a solution.

\[ y = 2e^{2x} + 3 \sin x - x \]
Finding the General Solution to the Nonhomogeneous Equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]

\[ y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x) \]

**Term:** Particular solution.
Finding a Particular Solution in the Constant Coefficient Case

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \]

**Term:** Undetermined coefficients, variation of parameters.
Example: Find the general solution to

\[ y^{(4)} - y = 2e^x + \cos x \]
Laplace Transforms

Motivation: Laplace transforms can be used to turn linear constant coefficient differential equations into algebraic equations.

**Definition** Let \( f \) be a continuous function on \([0, \infty)\). The Laplace transform of \( f \), denoted by \( \mathcal{L}[f(x)] \), or by \( F(s) \), is the function given by

\[
\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x) \, dx.
\]  

(1)

The domain of \( F \) is the set of all real numbers \( s \) for which the improper integral converges.
Illustrative Examples:

\[ L[f(x)] = \int_0^\infty e^{-sx} f(x) \, dx \]

\[ L[e^x] = \]

\[ L[e^{2x}] = \]

\[ L[e^{ax}] = \]
Recall: \[ L[f(x)] = \int_0^\infty e^{-sx} f(x) \, dx \]

As a result...

1. \[ L[\alpha f(x)] = \]

2. \[ L[f(x) + g(x)] = \]
\[ L [ y'(x) ] = \]

\[ L [ y''(x) ] = \]
Example:

Find the Laplace transform of the solution to

\[ y'' + 3y' - 4y = 3e^{-x} + 2, \quad y(0) = 2, \quad y'(0) = -1. \]

We will do this directly, without finding the solution first!!

Recall:

\[
L\left[ y'(x) \right] = -y(0) + sL\left[ y(x) \right]
\]

\[
L\left[ y''(x) \right] = -y'(0) - sy(0) + s^2 L\left[ y(x) \right]
\]
More Examples:

\[ L[1] = \]

\[ L[x] = \]

\[ L[\cos(x)] = \]

\[ L[\sin(x)] = \]
Table of Laplace Transforms

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<th>( f(x) )</th>
<th>( F(s) = \mathcal{L}[f(x)] )</th>
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<td>1</td>
<td>( \frac{1}{s} ), ( s &gt; 0 )</td>
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<td>( e^{\alpha x} )</td>
<td>( \frac{1}{s - \alpha} ), ( s &gt; \alpha )</td>
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<tr>
<td>( \cos \beta x )</td>
<td>( \frac{s}{s^2 + \beta^2} ), ( s &gt; 0 )</td>
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<td>( e^{\alpha x} \cos \beta x )</td>
<td>( \frac{s - \alpha}{(s - \alpha)^2 + \beta^2} ), ( s &gt; \alpha )</td>
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<td>( e^{\alpha x} \sin \beta x )</td>
<td>( \frac{\beta}{(s - \alpha)^2 + \beta^2} ), ( s &gt; \alpha )</td>
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<tr>
<td>( x^n, \ n = 1, 2, \ldots )</td>
<td>( \frac{n!}{s^{n+1}} ), ( s &gt; 0 )</td>
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<tr>
<td>( x^n e^{rx}, \ n = 1, 2, \ldots )</td>
<td>( \frac{n!}{(s - r)^{n+1}} ), ( s &gt; r )</td>
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<td>( x \cos \beta x )</td>
<td>( \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2} ), ( s &gt; 0 )</td>
</tr>
<tr>
<td>( x \sin \beta x )</td>
<td>( \frac{2\beta s}{(s^2 + \beta^2)^2} ), ( s &gt; 0 )</td>
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</table>

\( L\left[y'(x)\right] = -y(0) + sL\left[y(x)\right] \)

\( L\left[y''(x)\right] = -y'(0) - sy(0) + s^2L\left[y(x)\right] \)

\( L\left[f(x)\right] = \int_0^\infty e^{-sx} f(x) \, dx \)
\[ L[2\sin(x) - 3e^{-x} + 1] = \]

\[ L[3e^{2x}\cos(3x) + x\sin(x) - 3x^2] = \]
Note: $L[f(x)g(x)]$ is generally NOT equal to $L[f(x)]L[g(x)]$. 
**Example:** Use the Laplace transform to solve

\[ y'' - y' - 6y = e^{-x}, \quad y(0) = 2, \quad y'(0) = 3. \]

(Note: We can do this easier, without Laplace transforms, but I want to illustrate the process.)

**Recall:**

\[ L\left[y'(x)\right] = -y(0) + sL\left[y(x)\right] \]

\[ L\left[y''(x)\right] = -y'(0) - sy(0) + s^2 L\left[y(x)\right] \]
The TRUTH!!

The Laplace Transform is typically used to solve problems of the form

\[
\begin{align*}
  y'' + ay' + by &= f(t) \\
  y(0) &= b, \\n  y'(0) &= m
\end{align*}
\]

where \( f(t) \) is a piecewise defined function.