Vectors, Matrices and Systems of Equations

See Chapter 5 in the online text
and the related videos

Note: The midterm exam is scheduled in **301 AH**. Make sure you have used the online scheduler to select either Friday from 2-5pm or Saturday from 9-noon. A Laplace transform formula sheet will be provided. No notes or calculators will be permitted.

* **Open EMCF05a**

1. A

Systems of Linear Equations

2 equations
2 unknowns

2 equations
3 unknowns

$$\boxed{\begin{aligned} 2x-3y+z&=-1\\ x+y-3z&=2 \end{aligned}}$$

x, y, z

$$\boxed{\begin{aligned} 2x-5y&=1\\ x+3y&=0 \end{aligned}}$$

$$\boxed{\begin{aligned} 3x+2y-z&=1\\ 2x-y+2z&=0\\ -y+5z&=1 \end{aligned}}$$

3 equations
3 unknowns

x, y, z

3 equations
2 unknowns

$$\boxed{\begin{aligned} x-4y&=-1\\ -x+2y&=3\\ x-y&=2 \end{aligned}}$$

x, y

m equations
n unknowns

$$\boxed{\begin{aligned} a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,n}x_n&=b_1\\ a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,n}x_n&=b_2\\ &\vdots\\ a_{m,1}x_1+a_{m,2}x_2+\cdots+a_{m,n}x_n&=b_m \end{aligned}}$$

$a_{i,j}$, b_i known
x_j unknown.

What is a solution?

e.g.

$$\boxed{\begin{aligned} 3x+2y-z&=1\\ 2x-y+2z&=0\\ -y+5z&=1 \end{aligned}}$$

A solution is a "point" (x, y, z) that solves the system,

or a vector $\begin{pmatrix} x\\ y\\ z \end{pmatrix}$ that solves the system.

list a numbers.

Terminology

A system of linear equations is **consistent** if and only if

it has $\boxed{\text{a}}$ solution.

at least one

A system of linear equations is **inconsistent** if and only if

it has no solution.

(*) **The Truth:** A linear system of equations has either 0, 1 or infinitely many solutions.

0 solution

1 solution

Infinitely many solutions

All other cases boil down to this.

<u>Why?</u> Look at 2 equations with 2 unknowns.

<u>Generic</u>

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

2 Lines

a, b, c, d, e, f known real #s.

$$\boxed{\begin{array}{ll} \underline{\text{e.g.}} & 2x - 3y = 1 \\ & -x + \frac{1}{2}y = -2 \end{array}}$$

A solution will lie on both lines.

Hence, 0, 1 or ∞ sol'ns.

parallel and distinct

not parallel

same lines

The Matrix Form

($Au = b$, associated with a linear system)

$$\boxed{\begin{aligned}3x+2y-z&=1\\2x-y+2z&=0\\-y+5z&=1\end{aligned}} \leftrightarrow \underbrace{\begin{pmatrix}3&2&-1\\2&-1&2\\0&-1&5\end{pmatrix}}_{\text{coefficient matrix}}\underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}}_{\text{vector}}=\overbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}^{\text{vector}}$$

$$\boxed{\begin{matrix}a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,n}x_n=b_1\\a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,n}x_n=b_2\\\vdots\\a_{m,1}x_1+a_{m,2}x_2+\cdots+a_{m,n}x_n=b_m\end{matrix}} \leftrightarrow \overbrace{\begin{pmatrix}a_{1,1}&a_{1,2}&\cdots&a_{1,n}\\\vdots&&&\\a_{m,1}&a_{m,2}&\cdots&a_{m,n}\end{pmatrix}}\overbrace{\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}}=\overbrace{\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}}$$

The Augmented Matrix

(associated with a linear system)

rhs vector

$$\boxed{\begin{aligned} 3x+2y-z&=1\\ 2x-y+2z&=0\\ -y+5z&=1 \end{aligned}}$$

$$\left(\begin{array}{cccc} 3 & 2 & -1 & 1\\ 2 & -1 & 2 & 0\\ 0 & -1 & 5 & 1 \end{array}\right)$$

Coef. matrix

$$\boxed{\begin{array}{c} a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,n}x_n=b_1\\ a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,n}x_n=b_2\\ \vdots\\ a_{m,1}x_1+a_{m,2}x_2+\cdots+a_{m,n}x_n=b_m \end{array}}$$

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1\\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2\\ \vdots & \vdots & & \vdots & \vdots\\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_n \end{array}\right)$$

Using Row Operations To Solve Linear Systems

2 B

3 Elementary Row Operations:
(The basis for all modern computer code used to solve linear systems of equations.)

1. switch 2 rows (of the augmented matrix)

$R_i \leftrightarrow R_j$

corresponds with switching 2 equations

2. multiply a row by a nonzero scalar.

$\alpha R_i \to R_i$

$\alpha \neq 0$

corresponds to multiplying the corresponding equation by the scalar.

3. Add a multiple of one row to another row.

$\alpha R_i + R_j \to R_j$

$i \neq j$

add a multiple of one equation to another.

Elementary row operations preserve the solutions of a linear system of equations.

Example

Use elementary row operations to solve:

$$\begin{aligned} -3x+10y+4z&=1\\ -2x+7y+4z&=2\\ 4x-9y+11z&=15 \end{aligned}$$

$$\begin{pmatrix} -3 & 10 & 4 & 1 \\ -2 & 7 & 4 & 2 \\ 4 & -9 & 11 & 15 \end{pmatrix}$$

Goal: make the entries in the box zero.

How: work left to right.

Combine type 1 + 2:

$2R_1 - 3R_2 \to R_2$

$4R_1 + 3R_3 \to R_3$

$$\begin{pmatrix} -3 & 10 & 4 & 1 \\ 0 & -1 & -4 & -4 \\ 0 & \boxed{13} & 49 & 49 \end{pmatrix}$$

$13R_2 + R_3 \to R_3$

$$\begin{pmatrix} -3 & 10 & 4 & 1 \\ 0 & -1 & -4 & -4 \\ 0 & 0 & -3 & -3 \end{pmatrix} \begin{array}{l} \to -3x+10y+4z=1 \\ \to -y-4z=-4 \\ \to -3z=-3 \\ \quad \Rightarrow z=1 \end{array}$$

$-y-4=-4$

$y=0$

$-3x+0+4=1$

$\Rightarrow x=1$

$\therefore$ The sol'n is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

ONE Solution!

http://online.math.uh.edu Click on the link to the online matrix calculator.

The augmented matrix is

-3, 10, 4, 1;
-2, 7, 4, 2;
4, -9, 11, 15;

(2)R1 + (-3)R2 -> R2 gives

-3, 10, 4, 1;
0, -1, -4, -4;
4, -9, 11, 15;

(4)R1 + (3)R3 -> R3 gives

-3, 10, 4, 1;
0, -1, -4, -4;
0, 13, 49, 49;

(13)R2 + (1)R3 -> R3 gives

-3, 10, 4, 1;
0, -1, -4, -4;
0, 0, -3, -3;

$\Rightarrow \cdots \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Example

Use elementary row operations to solve:

$$-3x+10y+4z=1$$
$$-2x+7y+4z=2$$
$$x-3y=2$$

The augmented matrix is

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ -2, & 7, & 4, & 2; \\ 1, & -3, & 0, & 2; \end{pmatrix}$$

(2)R1 + (-3)R2 -> R2 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 1, & -3, & 0, & 2; \end{pmatrix}$$

(1)R1 + (3)R3 -> R3 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 0, & 1, & 4, & 7; \end{pmatrix}$$

(1)R2 + (1)R3 -> R3 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 0, & 0, & 0, & 3; \end{pmatrix} \rightarrow 0x+0y+0z=3 \quad \text{i.e.} \quad \underline{\underline{0=3}}$$

'impossible'

No Solution!

Example

Use elementary row operations to solve:

$$-3x + 10y + 4z = 1$$
$$-2x + 7y + 4z = 2$$
$$x - 3y = \boxed{1}$$

The augmented matrix is

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ -2, & 7, & 4, & 2; \\ 1, & -3, & 0, & 1; \end{pmatrix}$$

(2)R1 + (-3)R2 -> R2 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 1, & -3, & 0, & 1; \end{pmatrix}$$

(1)R1 + (3)R3 -> R3 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 0, & 1, & 4, & 4; \end{pmatrix}$$

(1)R2 + (1)R3 -> R3 gives

$$\begin{pmatrix} -3, & 10, & 4, & 1; \\ 0, & -1, & -4, & -4; \\ 0, & 0, & 0, & 0; \end{pmatrix}$$

$\to -3x + 10y + 4z = 1$

$\to -y - 4z = -4 \iff \boxed{y = 4 - 4z}$

$\to 0 = 0$. No problem!

$$-3x + 10(4 - 4z) + 4z = 1$$
$$-3x + 40 - 36z = 1$$
$$-3x = -39 + 36z$$
$$\boxed{x = 13 - 12z}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 - 12z \\ 4 - 4z \\ z \end{pmatrix}$$ where z is any real number!

∞ many solutions.

e.g. use $z = 1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

e.g. use $z = \frac{1}{2} \Rightarrow \begin{pmatrix} 7 \\ 2 \\ 1/2 \end{pmatrix}$

$\vdots$

Matrix Multiplication

3 B

Definition:

If A is an mxk matrix and B is a kxn matrix, then AB is defined, and the product AB is a mxn matrix. The entry in row i and column j of AB is given by the dot product of row i of A with column j of B.

??

e.g. Dot product of $(2\ -1\ \ 3)$ with $\begin{pmatrix} -4 \\ 6 \\ 7 \end{pmatrix}$ is $(2)(-4)+(-1)(6)+(3)(7)$
$= -8-6+21 = 7$

Matrix multiplication is very different from multiplying numbers!! We have to be careful.

Properties:

① AB is usually different from BA, even if both make sense.

② $A(B+C) = AB + AC$

You can add matrices if they have the same size. You add them term by term.

③ $\alpha AB = A(\alpha B)$ where α is a scalar

Special Matrices:

① Zero matrix – all terms are 0.

$$O_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Notes: i. If A is mxn then

$$A + O_{m\times n} = A$$

ii. If A is mxn then

$$A\,O_{n\times k} = O_{m\times k}$$

② Identity matrix (multiplicative identity)

$$I_{1\times 1} = 1, \quad I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{4\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice, there are ones on the diagonal, and zeros everywhere else.

If A is $m\times n$ then
$A\,I_{n\times n} = A$ and $I_{m\times m}A = A$

$\underline{\underline{4}}$ A

$$\begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{5} & \boxed{5} \\ \boxed{2} & \boxed{2} \end{pmatrix}$$

2×2

2×2 $\quad$ 2×2

$$\begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{2} & \boxed{3} \\ \boxed{-1} & \boxed{3} \end{pmatrix}$$

$$A=\begin{pmatrix}1 & -1\\ \boxed{2 \quad 3}\end{pmatrix}, B=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}, C=\begin{pmatrix}1 & 0 & 1\\ 2 & -1 & 1\end{pmatrix}, D=\begin{pmatrix}-1 & 1 & 0\\ 0 & 1 & 2\\ 2 & 1 & -1\end{pmatrix}$$

$2\times\underline{\underline{2}}$ $\quad$ 2×2 $\quad$ $\underline{\underline{2}}\times 3$

		Comments
$AB =$ ✓ 2×2	$\begin{pmatrix}1 & -2\\ 7 & 11\end{pmatrix}$	Different!
$BA =$ ✓ $2\times\underline{\underline{2}}$ $\underline{\underline{2}}\times 2$	$\begin{pmatrix}4 & 1\\ 7 & 8\end{pmatrix}$	
AC is 2×3	$\begin{pmatrix}-1 & 1 & 0\\ 8 & -3 & 5\end{pmatrix}$	
CA is	Not defined	$2\times\underline{3}$ times $\underline{2}\times 2$ is impossible
DC is	Not defined	$3\times\underline{\underline{3}}$ $\quad$ $\underline{\underline{2}}\times 3$ nope
CD is	yes! It is 2×3	$2\times\underline{3}$ $\quad$ $\underline{3}\times 3$

Invertible Matrices

For real #s

$x \cdot \frac{1}{x} = 1$

$x \neq 0$

Definition:

An nxn matrix A is invertible if and only if there is an nxn matrix B so that $AB = I_{nxn}$

Incredibly, in this case, it can be shown that we also get $BA = I_{nxn}$.

When this happens, the matrix B is referred to as the inverse of the matrix A, and we write $A^{-1} = B$.

notation

$A^{-1} \equiv$ the inverse of A.

Finding the inverse of a matrix is MUCH different than finding the inverse of a number. Also, there are nonzero matrices that do not have inverses!!

Spoiler Alert!!! The matrices that have inverses are exactly the matrices that have nonzero determinants.

Solving $Ax = b$ when A is invertible:

spse A is nxn and invertible.

Then $Ax = b$ iff

A: nxn known

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ unknown vector

b: nx1 known

$$\underbrace{A^{-1}A}x = A^{-1}b$$

It is important that you multiply both sides in the same manner.

$$\underbrace{I_{nxn}x} = A^{-1}b$$

$$x = A^{-1}b$$

the solution!

Computing the Inverse of an Invertible Matrix

Using RREF:

write

$$\begin{pmatrix} A & I_{n\times n} \end{pmatrix} \xrightarrow{\text{do elementary row operations}} \begin{pmatrix} I_{n\times n} & A^{-1} \end{pmatrix}$$

$n \times 2n$ matrix

This process is possible if and only if A is invertible.

In other words, A is invertible if and only if you can turn A into I_{nxn} using elementary row operations. AND, this is possible, if and only if you can use elementary row operations to get all zeros below the diagonal of A with nonzero entries on the diagonal of A.

Let $A = \begin{pmatrix} 1 & -4 & 3 \\ 1 & -5 & 4 \\ -1 & 1 & 1 \end{pmatrix}$. Determine if A is invertible, and if so, find A^{-1}.

$$\begin{pmatrix} 1 & -4 & 3 & 1 & 0 & 0 \\ 1 & -5 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Phase 1: Turn these to 0.

$$\begin{matrix} 1, & -4, & 3, & 1, & 0, & 0; \\ 1, & -5, & 4, & 0, & 1, & 0; \\ \underbrace{-1, \;\; 1, \;\; 1,}_{A} & & & \underbrace{0, \;\; 0, \;\; 1;}_{I_{3\times3}} & & \end{matrix}$$

(1)R1 + (-1)R2 -> R2 gives

1, -4, 3, 1, 0, 0;
0, 1, -1, 1, -1, 0;
-1, 1, 1, 0, 0, 1;

(1)R1 + (1)R3 -> R3 gives

1, -4, 3, 1, 0, 0;
0, 1, -1, 1, -1, 0;
0, -3, 4, 1, 0, 1;

(3)R2 + (1)R3 -> R3 gives

1, -4, 3, 1, 0, 0;
0, 1, -1, 1, -1, 0;
0, 0, 1, 4, -3, 1;

(1)R3 + (1)R2 -> R2 gives

1, -4, 3, 1, 0, 0;
0, 1, 0, 5, -4, 1;
0, 0, 1, 4, -3, 1;

(-3)R3 + (1)R1 -> R1 gives

1, -4, 0, -11, 9, -3;
0, 1, 0, 5, -4, 1;
0, 0, 1, 4, -3, 1;

(4)R2 + (1)R1 -> R1 gives

$$\begin{matrix} 1, \;\; 0, \;\; 0, & 9, \;\; -7, \;\; 1; \\ 0, \;\; 1, \;\; 0, & 5, \;\; -4, \;\; 1; \\ \underbrace{0, \;\; 0, \;\; 1,}_{I_{3\times3}} & \underbrace{4, \;\; -3, \;\; 1;}_{A^{-1}} \end{matrix}$$

Note:

the_Display_Matrix is now
1, -4, 3;
1, -5, 4;
-1, 1, 1;
The inverse of the_Display_Matrix is
9, -7, 1;
5, -4, 1;
4, -3, 1;

you can automate.

AND

the_Display_Matrix is now
1, -4, 3, 1, 0, 0;
1, -5, 4, 0, 1, 0;
-1, 1, 1, 0, 0, 1;
The rref of the_Display_Matrix is
1, 0, 0, 9, -7, 1;
0, 1, 0, 5, -4, 1;
0, 0, 1, 4, -3, 1;

5 C

Important: For large systems, you would never use the inverse of a matrix to solve the system. WHY? Because it takes the same number of computations to find the inverse of an nxn matrix as it takes to solve n systems (unless the matrix has some special structure).

In general, elementary row operations are always used to solve systems.

Example: **Solve** $\begin{pmatrix} 1 & -4 & 3 \\ 1 & -5 & 4 \\ -1 & 1 & 1 \end{pmatrix} x = b$ **for** $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Using an inverse matrix:

we just found $\begin{pmatrix} 1, & -4, & 3; \\ 1, & -5, & 4; \\ -1, & 1, & 1; \end{pmatrix}^{-1} = \begin{pmatrix} 9, & -7, & 1; \\ 5, & -4, & 1; \\ 4, & -3, & 1; \end{pmatrix}$

$$\begin{pmatrix} 1 & -4 & 3 \\ 1 & -5 & 4 \\ -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1, & -4, & 3; \\ 1, & -5, & 4; \\ -1, & 1, & 1; \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9, & -7, & 1; \\ 5, & -4, & 1; \\ 4, & -3, & 1; \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 3 \\ 1 & -5 & 4 \\ -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1, & -4, & 3; \\ 1, & -5, & 4; \\ -1, & 1, & 1; \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 9, & -7, & 1; \\ 5, & -4, & 1; \\ 4, & -3, & 1; \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ -4 \end{pmatrix}$$

you do the others...

The truth!!!! A random nxn matrix will have a VERY nasty looking inverse.

the_Display_Matrix is now
1, -1, -1;
-2, 8, -7;
-3, 5, 5;
The inverse of the_Display_Matrix is
5/2, 0, 1/2;
31/30, 1/15, 3/10;
7/15, -1/15, 1/5;
the_Display_Matrix is now
6, -8, -4;
-10, 6, 7;
-9, 10, 2;
The inverse of the_Display_Matrix is
-29/90, -2/15, -8/45;
-43/180, -2/15, -1/90;
-23/90, 1/15, -11/45;
the_Display_Matrix is now
7, -1, -5;
10, 7, 4;
8, 7, 7;
The inverse of the_Display_Matrix is
21/115, -28/115, 31/115;
-38/115, 89/115, -78/115;
14/115, -57/115, 59/115;
the_Display_Matrix is now
1, -7, 1;
-6, 10, 2;
-3, 1, -9;
The inverse of the_Display_Matrix is
-23/88, -31/176, -3/44;
-15/88, -3/176, -1/44;
3/44, 5/88, -1/11;
the_Display_Matrix is now
-2, 7, -7;
8, -7, 7;
-10, -8, -5;
The inverse of the_Display_Matrix is
1/6, 1/6, 0;
-5/91, -10/91, -1/13;
-67/273, -43/273, -1/13;
the_Display_Matrix is now
-6, 7, -2;
5, -9, -3;
-3, 9, -5;
The inverse of the_Display_Matrix is
-36/115, -17/230, 39/230;
-17/115, -12/115, 14/115;
-9/115, -33/230, -19/230;
the_Display_Matrix is now
-1, 4, -7;
10, 5, -7;
-1, 8, 3;
The inverse of the_Display_Matrix is
-71/758, 34/379, -7/758;
23/758, 5/379, 77/758;
-85/758, -2/379, 45/758;