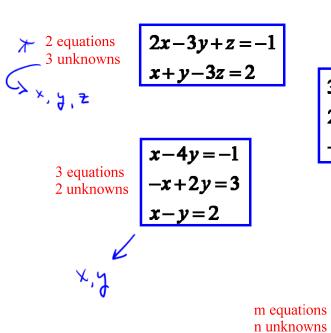
## **Vectors, Matrices and Systems of Equations**

See Chapter 5 in the online text and the related videos

**Note:** The midterm exam is scheduled in **110 CBB**. Make sure you have used the online scheduler to select either Friday from 2-5pm or Saturday from 9-noon. A Laplace transform formula sheet will be provided. No notes or calculators will be permitted.

**Open EMCF05** 

## Systems of Linear Equations



$$2x-3y+z=-1$$
$$x+y-3z=2$$

$$3x+2y-z=1$$

$$2x-y+2z=0$$

$$-y+5z-1$$

$$2x-5y=1$$
$$x+3y=0$$

$$x + 3y = 0$$

2 equations 2 unknowns

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$$

ai and by are known. Xi are unknown.

Question: What is a "solution" to one of these systems?

A collection of values for the equations unknowns that makes the equations ex. (x+y=1) (x-y=2)Add equs  $(x-3) \Rightarrow (x=\frac{3}{2})$   $(x-\frac{3}{2})$ 

$$\left(\begin{array}{c} x = \frac{3}{2} \\ y = -\frac{1}{2} \end{array}\right)$$

2

# **Terminology**

A system of linear equations is **consistent** if and only if

it has at least one solution

A system of linear equations is **inconsistent** if and only if

it does not have a solution.

The Truth: A linear system of equations has either 0, 1 or infinitely many solutions.

0 solution

1 solution

Infinitely many solutions

Why? It all boils down to 2 equations

and 2 unknowns.

a,b,c,d,e,f

dx + ey = f known.

A solution is a point of intersection.

Possibilities:

1. 2 non parallel lines.

2. 2 district parallel lines.

No solution

3. 2 identical parallel lines.

# The Matrix Form

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column vector

( Au = b, associated with a linear system)

$$-2x+3y+z=1$$
$$3x-2y+4z=-2$$
$$y-3z=6$$

$$\begin{pmatrix} -2 & 3 & 1 \\ 3 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} \times \\ 2 \\ \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ \end{pmatrix}$$

$$(hs)$$

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$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$$

$$\begin{vmatrix} a_{1,1} & a_{12} & \cdots & a_{1,n} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

$$M \times N$$

$$+ A KADMA$$

# The Augmented Matrix

(associated with a linear system)

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$

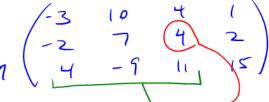
$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$$

$$-3x+10y+4z=1$$

$$-2x+7y+4z=2$$

$$4x - 9y + 11z = 15$$



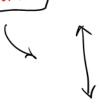
- (4). Give the (2,3) entry in the augmented matrix for the system.
- 2. Give the number of columns in the coefficient <u>matrix</u> for the system.

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# Using Row Operations To Solve Linear Systems

3 Elementary Row Operations: Operations on the argumented matrix (The basis for all modern computer code used to solve linear systems of equations.)

- 1. Multiply a row by a nonzero Scalar.

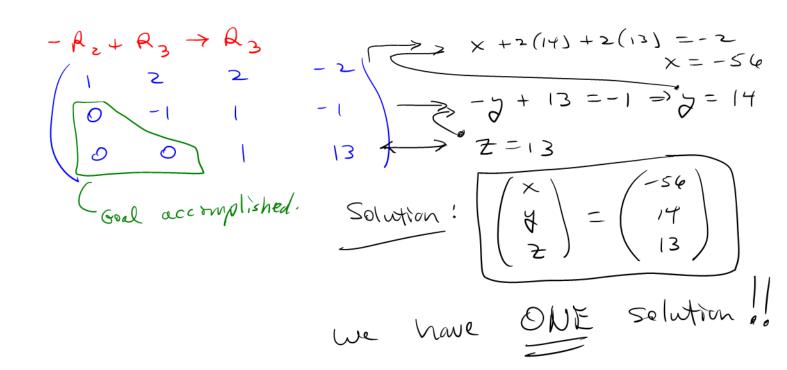
  (correst to multiplying an equation by the nonzero  $\propto R_i \rightarrow R_i$ ,  $\propto \mp 0$
- 2. Swop 2 rows R;  $\Leftrightarrow$  R; (corres. to swapping 2 equations.)
- 3. Replace row j with a scalar times Row i
  plus Row j.  $x R_1 + R_2 \rightarrow R_3$

Elementary row operations preserve the solutions of a linear system of equations.

# Example

Use elementary row operations to solve:

$$3x+5y+8z=6$$
$$-x-3y-z=1$$
$$x+2y+2z=-2$$



```
My augmented matrix is
3, 5, 8, 6;
-1, -3, -1, 1;
1, 2, 2, -2;
R1 <-> R2 gives
-1, -3, -1, 1;
3, 5, 8, 6;
1, 2, 2, -2;
(3)R1 + (1)R2 -> R2 gives
-1, -3, -1, 1;
0, -4, 5, 9;
1, 2, 2, -2;
(1)R1 + (1)R3 -> R3 gives
-1, -3, -1, 1;
0, -4, 5, 9;
0, -1, 1, -1;
R2 <-> R3 gives
-1, -3, -1, 1;
0, -1, 1, -1;
0, -4, 5, 9;
(-4)R2 + (1)R3 -> R3 gives
-1, -3, -1, 1;
0, -1, 1, -1;
0, 0, 1, 13;
(-1)R3 + (1)R2 -> R2 gives
-1, -3, -1, 1;
0, -1, 0, -14;
0, 0, 1, 13;
(1)R3 + (1)R1 -> R1 gives
-1, -3, 0, 14;
0, -1, 0, -14;
0, 0, 1, 13;
(-3)R2 + (1)R1 -> R1 gives
-1, 0, 0, 56;
0, -1, 0, -14;
0, 0, 1, 13;
(-1)R1 -> R1 gives
1, 0, 0, -56;
0, -1, 0, -14;
0, 0, 1, 13;
(-1)R2 -> R2 gives
1, 0, 0, -56;
0, 1, 0, 14;
0, 0, 1, 13;
```

Therefore, my solution is (x,y,z) = (-56,14,13).

We can also do this in one fell swoop with the RREF command. The augmented matrix is 3, 5, 8, 6; -1, -3, -1, 1; 1, 2, 2, -2; The rref of this matrix is 1, 0, 0, -56; 0, 1, 0, 14; 0, 0, 1, 13; Therefore, the solution is (x,y,z) = (-56,14,13).

## EMCF05

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \\ 1 & 1 & 6 \end{pmatrix}$$

- 3. What is the (2,3) entry in the matrix obtained by performing the elementary row operation  $-2R_1 + R_2 \longrightarrow R_2$  on the matrix A?
- 4. What is the (2,3) entry in the matrix obtained by performing the elementary row operation  $R_1 \longleftrightarrow R_2$  on the matrix A?
  - 5. What is the (2,3) entry in the matrix obtained by performing the elementary row operation  $3R_2 \longrightarrow R_2$  on the matrix A?

## Example

Use elementary row operations to solve:

$$3x + 5y + 8z = 6$$
$$-x - 3y - z = 1$$

$$2x + 2y + 7z = 7$$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 3 & 5 & 8 & 6 \\ 2 & 2 & 7 & 7 \end{pmatrix}$$

$$2R_1 + R_3 \rightarrow R_3$$

$$-A_2+A_3 \rightarrow A_3$$

$$-43+52=9$$
  $\Rightarrow y=\frac{5}{4}z-\frac{9}{4}$ 

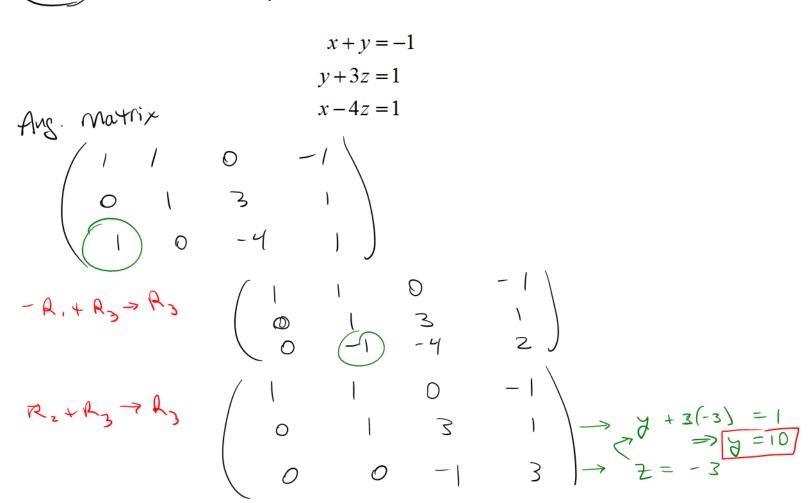
 $-\times$   $-3\left(\frac{5}{4}z^{2}-\frac{9}{4}\right)$  -z=1-x= 1572 - 27 + 2 + 1 Treat  $\pm 1$  i.e.  $x = -\frac{19}{4} \pm 1 \pm \frac{23}{4}$  a parameter  $\pm 1$ .  $\begin{pmatrix} \times \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{19}{4}t + \frac{23}{4} \\ \frac{5}{4}t - \frac{9}{4} \\ t \end{pmatrix}, \quad \text{total number.}$ Note:  $\begin{pmatrix} \times \\ 3 \end{pmatrix} = \pm \begin{pmatrix} -19/4 \\ 5/4 \end{pmatrix} + \begin{pmatrix} 23/4 \\ -9/4 \end{pmatrix}$ param for a line in 3-space Infinitely many solins. we get a different value of t.

## Example

3x + 5y + 8z = 6Use elementary row operations to solve: -x-3y-z=12x + 2y + 7z = 2 $\begin{pmatrix}
3 & 5 & 8 & 6 \\
-1 & -3 & -1 & 1 \\
2 & 2 & 7 & 2
\end{pmatrix}$   $\begin{pmatrix}
3 & 5 & 8 & 6 \\
3 & 2 & 7 & 2
\end{pmatrix}$  $3R_{1}+R_{2}\rightarrow R_{2}$   $2R_{1}+R_{3}\rightarrow R_{3}$  0 -4 5 9 -4 5 4  $-R_{2}+R_{3}\rightarrow R_{3}$  0 -4 5 9 0 -4 5 9

## EMCF05

6. Give the value of y associated with the solution to



#### EMCF05

Determine the value for h so that the system below is not consistent. I won't grade this.

$$x + y - 3z = -1$$

$$3x + 4y - hz = 2$$

8. Give the number of solutions to 
$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
.  $\begin{pmatrix} -\times_1 + 2 \times_2 = 1 \\ \times_1 + 3 \times_2 = 1 \end{pmatrix}$ 

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\frac{x_1 + 3x_2 = x_1 + 3x_2 = x$$

9. Give the number of solutions to

$$(x_1 + x_2 + 3x_3 = 4)$$
 $-x_1 + x_2 - x_3 = 3$ 
 $2x_1 + 2x_2 = 7$ 

$$\begin{pmatrix} x_3 \\ 1 \\ -3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & 9-b & 9-b \end{pmatrix}$$

Agron Says had is trouble. Let's see

$$7 + 7 = 7 = 7 = 7$$

Agron Says  $7 + 7 = 7 = 7$ 
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$y = (h-9) \pm +5$ $x + (h-9) \pm +5 -3 \pm = -1$ $x = (12-h) \pm -6$ $x $	
Matrix Multiplication \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	fe
Definitions:	_
Vectors:	
Dot product of two vectors:  Tuesday's  video	
If A is an mxk matrix and B is a kxn matrix, then AB is defined, and the product AB is a mxn matrix. The entry in row i and column i of	

If A is an mxk matrix and B is a kxn matrix, then AB is defined, and the product AB is a mxn matrix. The entry in row i and column j of AB is given by the dot product of row i of A with column j of B.

Properties:

Special Matrices: Zero and Identity

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

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**Examples:** 

## EMCF05

10. Give the (3,2) entry in the product

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 1 & -5 \end{pmatrix}$$

#### **Invertible Matrices**

#### Definition:

An nxn matrix A is invertible if and only if there is an nxn matrix B so that  $AB = I_{nxn}$ 

Incredibly, in this case, it can be shown that we also get  $BA = I_{nxn}$ .

See Yhe Video-When this happens, the matrix B is referred to as the inverse of the matrix A, and we write  $A^{-1} = B$ .

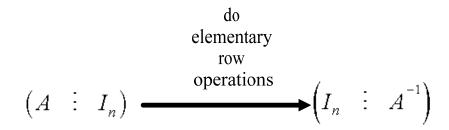
Finding the inverse of a matrix is MUCH different than finding the inverse of a number. Also, there are nonzero matrices that do not have inverses!!

Spoiler Alert!!! The matrices that have inverses are exactly the matrices that have nonzero determinants.

Solving Ax = b when A is invertible:

### Computing the Inverse of an Invertible Matrix A

#### Using RREF:



This process is possible if and only if A is invertible.

In other words, A is invertible if and only if you can turn A into  $I_{nxn}$  using elementary row operations. AND, this is possible, if and only if you can use elementary row operations to get all zeros below the diagonal of A with nonzero entries on the diagonal of A.

Determine if A is invertible, and if so, use  $A^{-1}$  to solve the linear system.

Determine if A is invertible, and if so, use  $A^{-1}$  to solve the linear system.

**Important:** For large systems, you would never use the inverse of a matrix to solve the system. WHY? Because it takes the same number of computations to find the inverse of an nxn matrix as it takes to solve n systems (unless the matrix has some special structure).

In general, elementary row operations are always used to solve systems.

The Truth!!!! A random nxn matrix will have a VERY nasty looking inverse.