Vectors, Matrices and Systems of Equations

See Chapter 5 in the online text and the related videos

Note: The midterm exam is scheduled in 110 CBB. Make sure you have used the online scheduler to select either Friday from 2-5pm or Saturday from 9-noon. A Laplace transform formula sheet will be provided. No notes or calculators will be permitted.

Open EMCF05
Systems of Linear Equations

2 equations
3 unknowns

\[
\begin{align*}
2x - 3y + z &= -1 \\
x + y - 3z &= 2
\end{align*}
\]

3 equations
2 unknowns

\[
\begin{align*}
x - 4y &= -1 \\
-x + 2y &= 3 \\
x - y &= 2
\end{align*}
\]

2 equations
2 unknowns

\[
\begin{align*}
2x - 5y &= 1 \\
x + 3y &= 0
\end{align*}
\]

3 equations
3 unknowns

\[
\begin{align*}
3x + 2y - z &= 1 \\
2x - y + 2z &= 0 \\
-y + 5z &= 1
\end{align*}
\]

m equations
n unknowns

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

\(a_{ij}\) and \(b_j\) are known. \(x_i\) are unknown.

**Question:** What is a "solution" to one of these systems?

**A:** A collection of values for the unknowns that makes the equations true.

\[\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]

ex.

\[
\begin{align*}
x + y &= 1 \\
x - y &= 2
\end{align*}
\]

\[\begin{align*}
2x &= 3 \\
y &= -\frac{1}{2}
\end{align*}\]

**Solution:** \(x = \frac{3}{2}, y = -\frac{1}{2}\) One solution.
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= \begin{pmatrix}
  3/2 \\
  -1/2
\end{pmatrix}
\]

Note: \[
\begin{pmatrix}
  x + y = 1 \\
  x - y = 2
\end{pmatrix}
\] \[\rightarrow\] \[
\begin{pmatrix}
  1 & 1 & 1 \\
  1 & -1 & 2
\end{pmatrix}
\]

Added \[\text{Row 1 and Row 2}\] \[
\rightarrow\] \[
\begin{pmatrix}
  1 & 1 & 1 \\
  2 & 0 & 3
\end{pmatrix}
\]

\[2x = 3\]

**Terminology**

A system of linear equations is **consistent** if and only if it has at least one solution.

A system of linear equations is **inconsistent** if and only if it does not have a solution.
The Truth: A linear system of equations has either 0, 1 or infinitely many solutions.

0 solution

1 solution

Infinitely many solutions

$$\begin{align*}
\text{Why? It all boils down to } & 2 \text{ equations and } 2 \text{ unknowns.} \\
\text{Generic: } & \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \\
\text{Geometrically: } & 2 \text{ lines.}
\end{align*}$$

A solution is a point of intersection.

Possibilities:

1. 2 non-parallel lines, one point of intersection.
2. 2 distinct parallel lines, no solution.
3. 2 identical parallel lines, infinitely many solutions.
The Matrix Form

( \( Au = b \), associated with a linear system)

\[
\begin{align*}
-2x + 3y + z &= 1 \\
3x - 2y + 4z &= -2 \\
y - 3z &= 6
\end{align*}
\]

\[
\begin{pmatrix}
-2 & 3 & 1 \\
3 & -2 & 4 \\
0 & 1 & -3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
-2 \\
0 \\
1
\end{pmatrix}
\]

General linear system

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots
\]

\[
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\]

\( m \times n \) unknown vector

\( k \times 1 \) rhs vector

\( m \times 1 \) rhs vector

\( k \times 1 \) unknown column vector

\( k \times 1 \) coefficient matrix for the system
The Augmented Matrix
(associated with a linear system)

\[-2x + 3y + z = 1\]
\[3x - 2y + 4z = -2\]
\[y - 3z = 6\]
1. Give the (2,3) entry in the augmented matrix for the system.

2. Give the number of columns in the coefficient matrix for the system.
Using Row Operations To Solve Linear Systems

3 Elementary Row Operations: Operations on the augmented matrix
(The basis for all modern computer code used to solve linear systems of equations.)

1. Multiply a row by a non-zero scalar.
   (corresponds to multiplying an equation by the non-zero scalar)
   \[ \alpha \cdot R_i \rightarrow R_i \quad \alpha \neq 0 \]

2. Swap 2 rows
   (corresponds to swapping 2 equations)
   \[ R_i \leftrightarrow R_j \]

3. Replace row j with a scalar times Row i plus Row j.
   \[ \alpha \cdot R_i + R_j \rightarrow R_j \]

Elementary row operations preserve the solutions of a linear system of equations.
Example

Use elementary row operations to solve:

\[ 3x + 5y + 8z = 6 \]
\[ -x - 3y - z = 1 \]
\[ x + 2y + 2z = -2 \]

1. Write the augmented matrix.
\[
\begin{pmatrix}
3 & 5 & 8 & 6 \\
-1 & -3 & -1 & 1 \\
1 & 2 & 2 & -2
\end{pmatrix}
\]

Strategy: Use ERO to obtain a form where you can solve for \( z \).

\[ R_1 \leftrightarrow R_3 \]
\[
\begin{pmatrix}
1 & 2 & 2 & -2 \\
-1 & -3 & -1 & 1 \\
3 & 5 & 8 & 6
\end{pmatrix}
\]

Use to kill \[ R_1 + R_2 \rightarrow R_2 \]
\[ -3R_1 + R_3 \rightarrow R_3 \]

\[
\begin{pmatrix}
1 & 2 & 2 & -2 \\
0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Now make this zero by using Row 2.
\[ -R_2 + R_3 \rightarrow R_3 \]

\[
\begin{pmatrix}
1 & 2 & 2 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

Goal accomplished.

\[
\begin{pmatrix}
-2 \\
-1 \\
13 \\
\end{pmatrix}
\]

\[
x + 2(14) + 2(12) = -2 \\
x = -56 \\
-\gamma + 13 = -1 \Rightarrow \gamma = 14 \\
\gamma = 13
\]

Solution:

\[
\begin{pmatrix}
x \\
\gamma \\
z \\
\end{pmatrix} = \begin{pmatrix}
-56 \\
14 \\
13 \\
\end{pmatrix}
\]

We have \textbf{ONE} solution!!
My augmented matrix is 

\[
\begin{align*}
3 & \quad 5 & \quad 8 & \quad 6 \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
1 & \quad 2 & \quad 2 & \quad -2 \\
R1 \leftrightarrow R2 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
3 & \quad 5 & \quad 6 & \quad 6 \\
1 & \quad 2 & \quad 2 & \quad -2 \\
(3)R1 + (1)R2 \rightarrow R2 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
0 & \quad -4 & \quad 5 & \quad 9 \\
1 & \quad 2 & \quad 2 & \quad -2 \\
(1)R1 + (1)R3 \rightarrow R3 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
0 & \quad -4 & \quad 5 & \quad 9 \\
0 & \quad -1 & \quad 1 & \quad -1 \\
R2 \leftrightarrow R3 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
0 & \quad -1 & \quad 1 & \quad -1 \\
0 & \quad -4 & \quad 5 & \quad 9 \\
(4)R2 + (1)R3 \rightarrow R3 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
0 & \quad -1 & \quad 1 & \quad -1 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
(1)R3 + (1)R2 \rightarrow R2 & \quad \text{gives} \\
-1 & \quad -3 & \quad -1 & \quad 1 \\
0 & \quad -1 & \quad 0 & \quad -14 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
(1)R3 + (1)R1 \rightarrow R1 & \quad \text{gives} \\
-1 & \quad -3 & \quad 0 & \quad 14 \\
0 & \quad -1 & \quad 0 & \quad -14 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
(3)R2 + (1)R1 \rightarrow R1 & \quad \text{gives} \\
-1 & \quad 0 & \quad 0 & \quad 56 \\
0 & \quad -1 & \quad 0 & \quad -14 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
(1)R1 \rightarrow R1 & \quad \text{gives} \\
1 & \quad 0 & \quad 0 & \quad -56 \\
0 & \quad -1 & \quad 0 & \quad -14 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
(1)R2 \rightarrow R2 & \quad \text{gives} \\
1 & \quad 0 & \quad 0 & \quad -56 \\
0 & \quad 1 & \quad 0 & \quad 14 \\
0 & \quad 0 & \quad 1 & \quad 13 \\
\end{align*}
\]

Therefore, my solution is \((x,y,z) = (-56, 14, 13)\).
\[ A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 0 & \boxed{-2} \\ 1 & 1 & 6 \end{pmatrix} \]

3. What is the (2,3) entry in the matrix obtained by performing the elementary row operation \(-2R_1 + R_2 \rightarrow R_2\) on the matrix \(A\)?

4. What is the (2,3) entry in the matrix obtained by performing the elementary row operation \(R_1 \leftrightarrow R_2\) on the matrix \(A\)?

5. What is the (2,3) entry in the matrix obtained by performing the elementary row operation \(3R_2 \rightarrow R_2\) on the matrix \(A\)?
Example

Use elementary row operations to solve:

\[
\begin{align*}
3x + 5y + 8z &= 6 \\
-x - 3y - z &= 1 \\
2x + 2y + 7z &= 7
\end{align*}
\]

Aug. Matrix:

\[
\begin{pmatrix}
3 & 5 & 8 & 6 \\
-1 & -3 & -1 & 1 \\
2 & 2 & 7 & 7
\end{pmatrix}
\]

\[R_2 \leftrightarrow R_1\]

\[
\begin{pmatrix}
-1 & -3 & -1 & 1 \\
3 & 5 & 8 & 6 \\
2 & 2 & 7 & 7
\end{pmatrix}
\]

\[\text{use } R_2 \rightarrow R_2 - 3R_1 \text{ and } 2R_1 + R_3 \Rightarrow R_3\]

\[
\begin{pmatrix}
-1 & -3 & -1 & 1 \\
0 & -4 & 5 & 9 \\
0 & -4 & 5 & 9
\end{pmatrix}
\]

\[-R_2 + R_3 \rightarrow R_3\]

\[
\begin{pmatrix}
-1 & -3 & -1 & 1 \\
0 & -4 & 5 & 9 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[b/c \ 0 = 0\]

Solve:

\[-4y + sz = 9 \quad \Rightarrow \quad y = \frac{5}{4} z - \frac{9}{4}\]
\[-x - 3 \left( \frac{5}{4} z - \frac{9}{4} \right) - z = 1 \]
\[-x = \frac{15}{4} z - \frac{27}{4} + z + 1 \]
\[\text{i.e.} \quad x = \frac{-19}{4} z + \frac{23}{4} \]

Solve:

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
-\frac{19}{4} t + \frac{23}{4} \\
\frac{5}{4} t - \frac{9}{4} \\
t
\end{pmatrix}, \quad t \text{ is a real number.}
\]

Note:

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= t \begin{pmatrix}
-\frac{19}{4} \\
\frac{5}{4} \\
1
\end{pmatrix}
+ \begin{pmatrix}
\frac{23}{4} \\
-\frac{9}{4} \\
0
\end{pmatrix}, \quad t \in \mathbb{R}
\]

Point on the line: \((x, y, z)\)
Direction vector for the line: \((\frac{-19}{4}, \frac{5}{4}, 1)\)
Param for a line in 3-space: \(t\)

Infinitely many solutions. We get a different solution for every different value of \(t\).
Example

Use elementary row operations to solve:

\[
\begin{align*}
3x + 5y + 8z &= 6 \\
-x - 3y - z &= 1 \\
2x + 2y + 7z &= 2
\end{align*}
\]

\[
\begin{pmatrix}
3 & 5 & 8 & 6 \\
-1 & -3 & -1 & 1 \\
2 & 2 & 7 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & -3 & -1 & 1 \\
3 & 5 & 8 & 6 \\
2 & 2 & 7 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & -3 & 1 \\
0 & -4 & 5 & 9 \\
0 & -4 & 5 & 4
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & -3 & 1 \\
0 & -4 & 5 & 9 \\
0 & 0 & 0 & -5
\end{pmatrix}
\]

Trouble! 0 = -5

No solution!!
6. Give the value of $y$ associated with the solution to

\[
x + y = -1
y + 3z = 1
x - 4z = 1
\]

Aug. Matrix

\[
\begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 3 & 1 \\
1 & 0 & -4 & 1
\end{pmatrix}
\]

$-R_1 + R_2 \rightarrow R_2$

$R_2 + R_3 \rightarrow R_3$

\[
\begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 0 & 3 & 1 \\
0 & 1 & 3 & 1 \\
0 & 0 & -1 & 3
\end{pmatrix}
\]

$y + 3(-3) = 1$  \[\Rightarrow y = 10\]
$z = -3$
7. Determine the value for $h$ so that the system below is not consistent.

$$x + y - 3z = -1$$
$$3x + 4y - hz = 2$$

8. Give the number of solutions to

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

9. Give the number of solutions to

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$ 

Aug. matrix

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -h & 2 \\ 3 \end{pmatrix}$$

$$-3R_1 + R_2 \rightarrow R_2$$

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & 9-h & 5 \end{pmatrix}$$

There is no value of $h$ that causes trouble!

Aaron says $h$ is trouble. Let's see $y + z = 5$ 

$$y = -z + 5$$ 

$$x + (-z + 5) - 3z = -1$$ 

$$x = 4z - 6$$
Matrix Multiplication

Definitions:

Vectors:

Dot product of two vectors:

If A is an mxk matrix and B is a kxn matrix, then AB is defined, and the product AB is a mxn matrix. The entry in row i and column j of AB is given by the dot product of row i of A with column j of B.

Properties:

Special Matrices: Zero and Identity
Aug. Matrix

\[
\begin{pmatrix}
1 & 1 & 3 & 4 \\
-1 & 1 & -1 & 3 \\
2 & 2 & 2 & 2
\end{pmatrix}
\]

\[r_1 + r_2 \rightarrow r_2\]

\[
\begin{pmatrix}
1 & 1 & 3 & 4 \\
0 & 2 & 2 & 7 \\
0 & 2 & 2 & 2
\end{pmatrix}
\]

\[
2y + 2z = 7 \\
2y + 2z = 2
\]

No Solution!
Examples:
10. Give the (3,2) entry in the product

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 1 & -5 \end{pmatrix}$$
Invertible Matrices

Definition:
An nxn matrix $A$ is invertible if and only if there is an nxn matrix $B$ so that $AB = I_{nxn}$

Incredibly, in this case, it can be shown that we also get $BA = I_{nxn}$.

When this happens, the matrix $B$ is referred to as the inverse of the matrix $A$, and we write $A^{-1} = B$.

Finding the inverse of a matrix is MUCH different than finding the inverse of a number. Also, there are nonzero matrices that do not have inverses!!

Spoiler Alert!!! The matrices that have inverses are exactly the matrices that have nonzero determinants.

Solving $Ax = b$ when $A$ is invertible:
Computing the Inverse of an Invertible Matrix \( A \)

Using RREF:

\[
\begin{align*}
&\text{do elementary row operations} \\
&\begin{pmatrix} A & I_n \end{pmatrix} \rightarrow \begin{pmatrix} I_n & A^{-1} \end{pmatrix}
\end{align*}
\]

This process is possible if and only if \( A \) is invertible.

In other words, \( A \) is invertible if and only if you can turn \( A \) into \( I_{nxn} \) using elementary row operations. AND, this is possible, if and only if you can use elementary row operations to get all zeros below the diagonal of \( A \) with nonzero entries on the diagonal of \( A \).
Let $A$ be the coefficient matrix for the linear system
\[
\begin{align*}
3x + 5y + 8z &= 6 \\
-x - 3y - z &= 1 \\
x + 2y + 2z &= -2
\end{align*}
\]

Determine if $A$ is invertible, and if so, use $A^{-1}$ to solve the linear system.
Let $A$ be the coefficient matrix for the linear system \[
\begin{pmatrix}
3x + 5y + 8z &= 6 \\
-x - 3y - z &= 1 \\
2x + 2y + 7z &= 7
\end{pmatrix}.
\]

Determine if $A$ is invertible, and if so, use $A^{-1}$ to solve the linear system.
**Important:** For large systems, you would never use the inverse of a matrix to solve the system. WHY? Because it takes the same number of computations to find the inverse of an $n \times n$ matrix as it takes to solve $n$ systems (unless the matrix has some special structure).

In general, elementary row operations are always used to solve systems.
The Truth!!!! A random $n \times n$ matrix will have a VERY nasty looking inverse.