Vectors, Matrices and Systems of Equations

See Chapter 5 in the online text
and the related videos

Note: The midterm exam is scheduled in **110 CBB**. Make sure you have used the online scheduler to select either Friday from 2-5pm or Saturday from 9-noon. A Laplace transform formula sheet will be provided. No notes or calculators will be permitted.

Open EMCF05

Systems of Linear Equations

2 equations
3 unknowns

x, y, z

$$\boxed{\begin{array}{l} 2x-3y+z=-1 \\ x+y-3z=2 \end{array}}$$

$$\boxed{\begin{array}{l} 3x+2y-z=1 \\ 2x-y+2z=0 \\ -y+5z=1 \end{array}}$$

3 equations
3 unknowns

x, y, z

$$\boxed{\begin{array}{l} 2x-5y=1 \\ x+3y=0 \end{array}}$$

2 equations
2 unknowns

x, y

3 equations
2 unknowns

$$\boxed{\begin{array}{l} x-4y=-1 \\ -x+2y=3 \\ x-y=2 \end{array}}$$

x, y

m equations
n unknowns

$$\boxed{\begin{array}{c} a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,n}x_n=b_1 \\ a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,n}x_n=b_2 \\ \vdots \\ a_{m,1}x_1+a_{m,2}x_2+\cdots+a_{m,n}x_n=b_m \end{array}}$$

$a_{i,j}$ and b_j are known. x_i are unknown.

Question: What is a "solution" to one of these systems?

A: A collection of values for the unknowns that makes the equations true.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

ex. $x+y=1$
$x-y=2$

Add eqns $\boxed{2x=3}$ $\Rightarrow x=\frac{3}{2}$ $\Rightarrow y=-\frac{1}{2}$

Solution: $\boxed{\begin{array}{l} x=3/2 \\ y=-1/2 \end{array}}$ One solution.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$$

$x + y = 1$

Note: $\begin{pmatrix} x + y = 1 \\ x - y = 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow[\text{Row 1 and Row 2}]{\text{Added}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix}$

$2x = 3$

Terminology

A system of linear equations is **consistent** if and only if

it has at least one solution.

A system of linear equations is **inconsistent** if and only if

it does not have a solution.

$\boxed{\textbf{The Truth:}}$ A linear system of equations has either 0, 1 or infinitely many solutions.

0 solution

1 solution

Infinitely many solutions

Why? It all boils down to 2 equations and 2 unknowns.

Generic:

$$\begin{pmatrix} ax + by = c \\ dx + ey = f \end{pmatrix}$$

a, b, c, d, e, f known.

Geometrically: 2 lines.

A solution is a point of intersection.

Possibilities:

1. 2 non-parallel lines.
   One point of intersection.

2. 2 distict parallel lines.
   No solution

3. 2 identical parallel lines
   Infinitely many sol'ns.

The Matrix Form

($Au = b$, associated with a linear system)

$$
\begin{aligned}
-2x + 3y + z &= 1 \\
3x - 2y + 4z &= -2 \\
y - 3z &= 6
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} -2 & 3 & 1 \\ 3 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix}}_{\text{coef. matrix for the system}}
\overbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}^{\text{unknown column vector}}
=
\underbrace{\begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}}_{\text{rhs vector}}
$$

General linear system

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}}_{m \times n}
\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{unknown vector}}
=
\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\text{rhs vector}}
$$

The Augmented Matrix

(associated with a linear system)

$$\boxed{\begin{aligned} -2x+3y+z&=1 \\ 3x-2y+4z&=-2 \\ y-3z&=6 \end{aligned}} \longleftrightarrow \begin{pmatrix} -2 & 3 & 1 & 1 \\ 3 & -2 & 4 & -2 \\ 0 & 1 & -3 & 6 \end{pmatrix}$$

$$\boxed{\begin{gathered} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{gathered}} \qquad \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_n \end{array}\right)$$

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$$-3x+10y+4z=1$$
$$-2x+7y+4z=2$$
$$4x-9y+11z=15$$

$$\begin{pmatrix} -3 & 10 & 4 & 1 \\ -2 & 7 & 4 & 2 \\ 4 & -9 & 11 & 15 \end{pmatrix}$$

4 1. Give the $\boxed{(2,3)}$ entry in the augmented matrix for the system.

3 2. Give the number of columns in the coefficient matrix for the system.

entry in the 2^{nd} $\boxed{\text{row}}$ and 3^{rd} $\boxed{\text{column}}$.

Using Row Operations To Solve Linear Systems

3 Elementary Row Operations: Operations on the augmented matrix
(The basis for all modern computer code used to solve linear systems of equations.)

1. multiply a row by a nonzero scalar.
(corres. to multiplying an equation by the nonzero scalar) $\alpha R_i \rightarrow R_i$, $\alpha \neq 0$

2. Swap 2 rows $R_i \leftrightarrow R_j$
(corres. to swapping 2 equations.)

3. Replace row j with a scalar times Row i plus Row j. $\alpha R_i + R_j \rightarrow R_j$

Elementary row operations preserve the solutions of a linear system of equations.

Example

Use elementary row operations to solve:

$$3x + 5y + 8z = 6$$
$$-x - 3y - z = 1$$
$$x + 2y + 2z = -2$$

1. write the augmented matrix.

$$\begin{pmatrix} 3 & 5 & 8 & 6 \\ -1 & -3 & -1 & 1 \\ 1 & 2 & 2 & -2 \end{pmatrix}$$

Strategy : use ERO to obtain

$$\begin{pmatrix} ? & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & 0 & ? & ? \end{pmatrix}$$

$\neq 0$ solve for z

$= 0$ can't solve for z.

$R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 1 & 2 & 2 & -2 \\ -1 & -3 & -1 & 1 \\ 3 & 5 & 8 & 6 \end{pmatrix}$$

use to kill

$R_1 + R_2 \to R_2$

$-3R_1 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & \boxed{-1} & 2 & 12 \end{pmatrix}$$

* Now make this zero by using Row 2.

$-R_2 + R_3 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 13 \end{array}\right)$$

Goal accomplished.

$x + 2(14) + 2(13) = -2$

$x = -56$

$-y + 13 = -1 \Rightarrow y = 14$

$z = 13$

Solution: $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -56 \\ 14 \\ 13 \end{pmatrix}}$

We have <u>ONE</u> solution!!

Matrix Calculator: http://online.math.uh.edu Click on "Matrix Calculator"

My augmented matrix is
3, 5, 8, 6;
-1, -3, -1, 1;
1, 2, 2, -2;
R1 <-> R2 gives
-1, -3, -1, 1;
3, 5, 8, 6;
1, 2, 2, -2;
(3)R1 + (1)R2 -> R2 gives
-1, -3, -1, 1;
0, -4, 5, 9;
1, 2, 2, -2;
(1)R1 + (1)R3 -> R3 gives
-1, -3, -1, 1;
0, -4, 5, 9;
0, -1, 1, -1;
R2 <-> R3 gives
-1, -3, -1, 1;
0, -1, 1, -1;
0, -4, 5, 9;
(-4)R2 + (1)R3 -> R3 gives
-1, -3, -1, 1;
0, -1, 1, -1;
0, 0, 1, 13;
(-1)R3 + (1)R2 -> R2 gives
-1, -3, -1, 1;
0, -1, 0, -14;
0, 0, 1, 13;
(1)R3 + (1)R1 -> R1 gives
-1, -3, 0, 14;
0, -1, 0, -14;
0, 0, 1, 13;
(-3)R2 + (1)R1 -> R1 gives
-1, 0, 0, 56;
0, -1, 0, -14;
0, 0, 1, 13;
(-1)R1 -> R1 gives
1, 0, 0, -56;
0, -1, 0, -14;
0, 0, 1, 13;
(-1)R2 -> R2 gives
1, 0, 0, -56;
0, 1, 0, 14;
0, 0, 1, 13;
Therefore, my solution is (x,y,z) = (-56,14,13).

We can also do this in one fell swoop
with the RREF command.
The augmented matrix is
3, 5, 8, 6;
-1, -3, -1, 1;
1, 2, 2, -2;
The rref of this matrix is
1, 0, 0, -56;
0, 1, 0, 14;
0, 0, 1, 13;
Therefore, the solution is
(x,y,z) = (-56,14,13).

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$$A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 0 & \boxed{-2} \\ 1 & 1 & 6 \end{pmatrix}$$

-8

3. What is the (2,3) entry in the matrix obtained by performing the elementary row operation $-2R_1 + R_2 \rightarrow R_2$ on the matrix A ?

3

4. What is the (2,3) entry in the matrix obtained by performing the elementary row operation $R_1 \leftrightarrow R_2$ on the matrix A ?

-6

5. What is the (2,3) entry in the matrix obtained by performing the elementary row operation $3R_2 \rightarrow R_2$ on the matrix A ?

Example

Use elementary row operations to solve:

$$3x + 5y + 8z = 6$$
$$-x - 3y - z = 1$$
$$2x + 2y + 7z = 7$$

Aug. Matrix:

$$\begin{pmatrix} 3 & 5 & 8 & 6 \\ -1 & -3 & -1 & 1 \\ 2 & 2 & 7 & 7 \end{pmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 3 & 5 & 8 & 6 \\ 2 & 2 & 7 & 7 \end{pmatrix}$$

use R_1 to zero-out 3 and 2.

$3R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 0 & -4 & 5 & 9 \\ 0 & -4 & 5 & 9 \end{pmatrix}$$ ← The same

$-R_2 + R_3 \to R_3$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 0 & -4 & 5 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ ← ok b/c $0 = 0$.

Solve:

$-4y + 5z = 9 \Rightarrow \boxed{y = \frac{5}{4}z - \frac{9}{4}}$

$$-x - 3\left(\tfrac{5}{4}z - \tfrac{9}{4}\right) - z = \underline{1}$$

$$-x = \tfrac{15}{4}z - \tfrac{27}{4} + z + \underline{1}$$

Treat z like a parameter t.

i.e. $\boxed{x = \frac{-19}{4}z + \frac{23}{4}}$

Sol'n: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{19}{4}t + \frac{23}{4} \\ \frac{5}{4}t - \frac{9}{4} \\ t \end{pmatrix}$, t is a real number.

$t \in \mathbb{R}$.

Note: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\underbrace{\begin{pmatrix} -19/4 \\ 5/4 \\ 1 \end{pmatrix}}_{\text{direction vector for the line}} + \underbrace{\begin{pmatrix} 23/4 \\ -9/4 \\ 0 \end{pmatrix}}_{\text{point on the line}}$

(param for $\boxed{\text{a line}}$ in 3-space

Infinitely many sol'ns. We get a different solution for every different value of t.

Example

Use elementary row operations to solve:

$$3x+5y+8z=6$$
$$-x-3y-z=1$$
$$2x+2y+7z=\boxed{2}$$

$$\begin{pmatrix} 3 & 5 & 8 & 6 \\ -1 & -3 & -1 & 1 \\ 2 & 2 & 7 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -3 & -1 & 1 \\ 3 & 5 & 8 & 6 \\ 2 & 2 & 7 & 2 \end{pmatrix}$$

$3R_1+R_2 \to R_2$

$2R_1+R_3 \to R_3$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 0 & -4 & 5 & 9 \\ 0 & -4 & 5 & 4 \end{pmatrix}$$

$-R_2+R_3 \to R_3$

$$\begin{pmatrix} -1 & -3 & -1 & 1 \\ 0 & -4 & 5 & 9 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

Trouble! $0=-5$

NO Solution!!

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6. Give the value of y associated with the solution to

$$x + y = -1$$
$$y + 3z = 1$$
$$x - 4z = 1$$

Aug. Matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & -4 & 1 \end{pmatrix}$$

$-R_1 + R_3 \to R_3$ $$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -4 & 2 \end{pmatrix}$$

$R_2 + R_3 \to R_3$ $$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

$\to y + 3(-3) = 1$

$\Rightarrow \boxed{y = 10}$

$\to z = -3$

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7. Determine the value for h so that the system below is not consistent.

Sorry :) I won't grade this.

→ no solution

Use 999 for infinity.

$$x + y - 3z = -1$$
$$3x + 4y - hz = 2$$

8. Give the number of solutions to $\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

1

$-x_1 + 2x_2 = 1$
$x_1 + 3x_2 = 1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

9. Give the number of solutions to

0

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

$x_1 + x_2 + 3x_3 = 4$
$-x_1 + x_2 - x_3 = 3$
$2x_2 + 2x_3 = 2$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Aug. matrix $\begin{pmatrix} 1 & 1 & -3 & -1 \\ \boxed{3} & 4 & -h & 2 \end{pmatrix}$

$-3R_1 + R_2 \to R_2$ $\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & 9-h & 5 \end{pmatrix}$

There is no value of h that causes trouble!

Aaron says ~~$h \neq 8$~~ is trouble. Let's see

$y + z = 5$ $y = -z + 5$

$x + (-z+5) - 3z = -1$
$x = 4z - 6$

inf. sol'ns. seems ok.

$y = (h-9)z + 5$

$x + [(h-9)z+5] - 3z = -1$

$x = (12-h)z - 6$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (12-h)z - 6 \\ (h-9)z+5 \\ z \end{pmatrix}$$

Inf. many sol'ns regardless of the value of h.

Matrix Multiplication

Definitions:

Vectors:

Dot product of two vectors:

See Tuesday's video

If A is an mxk matrix and B is a kxn matrix, then AB is defined, and **the product AB** is a mxn matrix. The entry in row i and column j of AB is given by the dot product of row i of A with column j of B.

Properties:

Special Matrices: Zero and Identity

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$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

Aug. Matrix

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ \boxed{-1} & 1 & -1 & 3 \\ \boxed{0} & 2 & 2 & 2 \end{pmatrix}$$

$R_1 + R_2 \to R_2$

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 2 & 2 & 7 \\ 0 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \boxed{\begin{array}{l} 2y + 2z = 7 \\ 2y + 2z = 2 \end{array}}$$

No solution!

Examples:

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10. Give the (3,2) entry in the product

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -5 \end{pmatrix}$$

Invertible Matrices

Definition:

An nxn matrix A is invertible if and only if there is an nxn matrix B so that $AB = I_{nxn}$

See the video.

Incredibly, in this case, it can be shown that we also get $BA = I_{nxn}$.

When this happens, the matrix B is referred to as the inverse of the matrix A, and we write $A^{-1} = B$.

Finding the inverse of a matrix is MUCH different than finding the inverse of a number. Also, there are nonzero matrices that do not have inverses!!

Spoiler Alert!!! The matrices that have inverses are exactly the matrices that have nonzero determinants.

Solving $Ax = b$ when A is invertible:

Computing the Inverse of an Invertible Matrix A

Using RREF:

$$\left(A \;\vdots\; I_n\right) \xrightarrow{\text{do elementary row operations}} \left(I_n \;\vdots\; A^{-1}\right)$$

This process is possible if and only if A is invertible.

In other words, A is invertible if and only if you can turn A into I_{nxn} using elementary row operations. AND, this is possible, if and only if you can use elementary row operations to get all zeros below the diagonal of A with nonzero entries on the diagonal of A.

Let A be the coefficient matrix for the linear system $\begin{pmatrix} 3x+5y+8z=6 \\ -x-3y-z=1 \\ x+2y+2z=-2 \end{pmatrix}$.

Determine if A is invertible, and if so, use A^{-1} to solve the linear system.

Let A be the coefficient matrix for the linear system $\begin{pmatrix} 3x+5y+8z=6 \\ -x-3y-z=1 \\ 2x+2y+7z=7 \end{pmatrix}$.

Determine if A is invertible, and if so, use A^{-1} to solve the linear system.

Important: For large systems, you would never use the inverse of a matrix to solve the system. WHY? Because it takes the same number of computations to find the inverse of an nxn matrix as it takes to solve n systems (unless the matrix has some special structure).

In general, elementary row operations are always used to solve systems.

The Truth!!!! A random nxn matrix will have a VERY nasty looking inverse.