Recall: Linear Systems of Equations

$Ax = b$

A: $m \times n$; x: $n \times 1$ unknown; b: $m \times 1$ known

1. Augmented Matrix

$$\begin{pmatrix} A & b \end{pmatrix} \quad m \times (n+1).$$

2. Elementary Row Operations

(i) $R_i \leftrightarrow R_j$ for $i \neq j$

(ii) $\alpha R_i \to R_i$ for $\alpha \neq 0$

(iii) $\alpha R_i + R_j \to R_j$ for $i \neq j$.

3. How do we undo elementary row operations?

swap back.

$\frac{1}{\alpha} R_i \to R_i$ for $\alpha \neq 0$

$-\alpha R_i + R_j \to R_j$ for $i \neq j$

i.e. elem row ops can be used to undo elem row ops.

Row Reduced Echelon Form

$\boxed{\text{(RREF)}}$

Motivating Example:

$$2x_1+3x_2+3x_3-x_4=2$$
$$x_1-2x_2+4x_3+x_4=-1$$
$$-x_1+3x_3+2x_4=1$$

3 equations
4 unknowns

Augmented matrix

$$\begin{pmatrix} 2 & 3 & 3 & -1 & 2 \\ 1 & -2 & 4 & 1 & -1 \\ -1 & 0 & 3 & 2 & 1 \end{pmatrix}$$

2, 3, 3, -1, 2;
1, -2, 4, 1, -1;
-1, 0, 3, 2, 1;
R1 <-> R2 gives
1, -2, 4, 1, -1;
2, 3, 3, -1, 2;
-1, 0, 3, 2, 1;
(-2)R1 + (1)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
-1, 0, 3, 2, 1;
(1)R1 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
0, -2, 7, 3, 0;
(1/7)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, -2, 7, 3, 0;
(2)R2 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;
(7/39)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;

you can solve from here

→ write the equations corresp. to these rows to get the solution.

1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;
(2)R2 + (1)R1 -> R1 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;
(5/7)R3 + (1)R2 -> R2 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;
(-18/7)R3 + (1)R1 -> R1 gives
1, 0, 0, -11/13, -5/13;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;

Easier to work with

$$x_1 - \frac{11}{13}x_4 = -\frac{5}{13}$$
$$x_2 - \frac{2}{13}x_4 = \frac{28}{39}$$
$$x_3 + \frac{5}{13}x_4 = \frac{8}{39}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{11}{13}x_4 - \frac{5}{13} \\ \frac{2}{13}x_4 + \frac{28}{39} \\ -\frac{5}{13}x_4 + \frac{8}{39} \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{11}{13} \\ \frac{2}{13} \\ -\frac{5}{13} \\ 1 \end{pmatrix} x_4 + \begin{pmatrix} -5/13 \\ 28/39 \\ 8/39 \\ 0 \end{pmatrix}$$

Infinitely many sol'ns.
You get a sol'n for each choice of x_4.

I used the Matrix Calculator linked from http://online.math.uh.edu

The augmented matrix is
2, 3, 3, -1, 2;
1, -2, 4, 1, -1;
-1, 0, 3, 2, 1;
R1 <-> R2 gives
1, -2, 4, 1, -1;
2, 3, 3, -1, 2;
-1, 0, 3, 2, 1;
(-2)R1 + (1)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
-1, 0, 3, 2, 1;
(1)R1 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
0, -2, 7, 3, 0;
(1/7)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, -2, 7, 3, 0;
(2)R2 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;

(2)R2 + (1)R1 -> R1 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;
(7/39)R3 -> R3 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;
(5/7)R3 + (1)R2 -> R2 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;
(-18/7)R3 + (1)R1 -> R1 gives
1, 0, 0, -11/13, -5/13;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;

Row Reduced Echelon Form (RREF)

General Process: Apply elementary row operations to a matrix until

1. The first nonzero entry in each row (if there is one) is 1.
   (These are called leading entries.)
2. The leading entry in a row is to the right of a leading entry in a row above.
3. The entries above and below leading entries are all 0.

Examples: Which of the following augmented matrices are in RREF?

$$\left(\begin{array}{ccccc} 1 & 3 & 3 & -1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad ? \quad \text{No}$$

needs to be 0.

leading entries

$$\left(\begin{array}{ccccc} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad ? \quad \text{No}$$

needs to be 0.

leading entries

$$\left(\begin{array}{cccc} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad ? \quad \text{Yes}$$

Example:

The RREF of the augmented matrix for the linear system

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 0 \\ 2x_1 - 9x_2 - 6x_3 - 3x_4 &= 1 \\ x_1 - 4x_2 - 2x_3 + x_4 &= -4 \end{aligned}$$

is

$$* \begin{pmatrix} 1 & 0 & 0 & 27 & -64 \\ 0 & 1 & 0 & 7 & -17 \\ 0 & 0 & 1 & -1 & 4 \end{pmatrix} *$$

$$\begin{aligned} x_1 + 27x_4 &= -64 \\ x_2 + 7x_4 &= -17 \\ x_3 - x_4 &= 4 \end{aligned}$$

Solve the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -27 \\ -7 \\ 1 \\ 1 \end{pmatrix} x_4 + \begin{pmatrix} -64 \\ -17 \\ 4 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= -27x_4 - 64 \\ x_2 &= -7x_4 - 17 \\ x_3 &= x_4 + 4 \\ x_4 &= x_4 \end{aligned}$$

Infinitely many sol'ns.

You get a different sol'n for each choice of x_4.

ex. $x_4 = 1 \Rightarrow (-91, -24, 5, 1)$

$x_4 = 2 \Rightarrow (-118, -31, 6, 2)$

$\vdots$

Computing the Inverse of an Invertible Matrix Using RREF

Process: If A is $n \times n$

1. $\begin{pmatrix} A & I_n \end{pmatrix}$

2. RREF this matrix

$\begin{pmatrix} \boxed{?} & B \end{pmatrix}$

If this is I_n then $B = A^{-1}$

If this is <u>not</u> I_n then A does not have an inverse.

Question

$m \times n$ matrices

$n \times 1$ matrices $\equiv$ column vector with n entries

Let $A \in R^{m,n}$ and $x \in R^{n}$. Denote the columns of A by $\vec{c}_1, \ldots, \vec{c}_n$ and suppose $x = (x_i)$. What is Ax?

$$A = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix} \qquad = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Columns of A

Linear combination of $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$

$$Ax = \vec{c}_1 \cdot x_1 + \vec{c}_2 x_2 + \cdots + \vec{c}_n \cdot x_n = \underbrace{x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_n \vec{c}_n}$$

Key phrase: linear combination.

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 2 & 1 \\ 1 & 2 & -7 \end{pmatrix} \text{ and } x = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix}. \quad \text{Then}$$

$$Ax = \begin{pmatrix} 2\cdot 1 + 4\cdot 2 + 0\cdot(-6) \\ 3\cdot 1 + 2\cdot 2 + 1\cdot(-6) \\ 1\cdot 1 + 2\cdot 2 + -7\cdot(-6) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\cdot 1 + \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}\cdot 2 + \begin{pmatrix} 0 \\ 1 \\ -7 \end{pmatrix}\cdot(-6)$$

$$= \vec{c}_1 \cdot x_1 + \vec{c}_2 \cdot x_2 + \vec{c}_3 \cdot x_3$$

Important Note: You can only solve $Ax = \vec{b}$ if $\vec{b}$ is a linear combination of the columns of A.

<u>Why?</u> Hm... $A = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Then $Ax = \vec{b}$ is the same as

$$\underbrace{x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n}_{\text{linear combination of } \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n} = \vec{b}$$

<u>Q</u>: What if $\vec{b} = \vec{0}$?

<u>A</u>: $x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n = \vec{0}$

<u>i.e.</u> a linear combination of $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ is $\vec{0}$.

<u>No duh!</u> $x_1 = x_2 = \cdots = x_n = 0$ works.

This is called the trivial linear combination.

Give a linear combination of

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ that gives $\vec{0}$.

i.e. Find x_1 and x_2 so that

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Easy! Use $x_1 = x_2 = 0$.

i.e. we can do it trivially.

Homogeneous vs Nonhomogeneous Linear Systems

$Ax = b$ is

nonhomogeneous if and only if $b \neq \vec{0}$

and homogeneous if and only if

$b = \vec{0}$.

Note: $A x = \vec{0}$ always has at least one sol'n. Just take $x = \vec{0}$. This is the trivial sol'n.

So $Ax = \vec{0}$ either has 1 sol'n (i.e. only $x = \vec{0}$) or infinitely many sol'ns.

The only linear comb. of the column vectors that is zero is the trivial one.

There are nontrivial linear combinations of the column vectors that give $\vec{0}$.

Definition

Let $\vec{c}_1, \ldots, \vec{c}_n \in R^m$. The set $\{\vec{c}_1, \ldots, \vec{c}_n\}$ is linearly independent if and only if $\boxed{x_1 \vec{c}_1 + \ldots + x_n \vec{c}_n = \vec{0},}$ with $x_i \in R$, implies $x_i = 0$ for all $i = 1, \ldots, n$. Otherwise, we say the set is linearly dependent.

i.e. the only linear comb that gives $\vec{0}$ is the trivial one. Equivalently, the only sol'n to

$$\begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix} x = \vec{0}$$

is $x = \vec{0}$.

Question

How is linear independence related to solving $Ax = 0$?

The columns of A are Linearly Independent iff the only sol'n is $x = \vec{0}$.

Example

Determine whether the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\}$ is linearly independent.

Sol'n: Solve $\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Aug matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$

$-R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

$x_1 - 0 = 0$

$\Rightarrow x_1 = 0$

$2x_2 = 0$

i.e.

$x_1 = x_2 = 0$.

$\therefore$ the vectors are linearly independent.

Example

Determine whether the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

Same drill:

Solve $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

1, -1, 1, 0;
1, 1, 3, 0;
2, -3, 1, 0;

Aug matrix

The rref is

1, 0, 2, 0;
0, 1, 1, 0;
0, 0, 0, 0;

$\rightarrow$ $x_1 + 2x_3 = 0$, $x_2 + x_3 = 0$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ -x_3 \\ x_3 \end{pmatrix}$

∞ many sol'ns. One for each choice of x_3.

$\Rightarrow$ the vectors are linearly dependent.

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1. Determine whether the set $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

is linearly independent or linearly dependent.

a. linearly independent

b. linearly dependent

c. there is not enough information

Solve $\begin{pmatrix} -1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow x_1 = x_2 = x_3 = 0.$ $\therefore$ Lin. Ind.

Determinants

How do we take the determinant of a 1x1 matrix?

$$A = (\alpha) \Rightarrow \det(A) = \alpha$$

How do we take the determinant of a 2x2 matrix?

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(A) = ad - bc$$

How do we take the determinant of an *n*x*n* matrix?

more complicated.

1. Notation: If A is $n \times n$ and $1 \le i, j \le n$ then $A_{i,j}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j.

Ex. $A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & 6 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow A_{3,1} = \begin{pmatrix} -2 & 1 \\ 1 & 6 \end{pmatrix}$

$A_{2,3} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$, $A_{1,1} = \begin{pmatrix} 1 & 6 \\ 1 & 4 \end{pmatrix}$

These methods all produce the same value. They are all called the determinant of the nxn matrix A.

Expand across any row. Ex. the i^{th} row

$$\det(A) = \sum_{k=1}^{n} a_{i,k} \cdot (-1)^{i+k} \det(A_{i,k})$$

Expand down any column. Ex. the j^{th} column.

$$\det(A) = \sum_{k=1}^{n} a_{k,j} \cdot (-1)^{k+j} \det(A_{k,j})$$

Expand down Column 3.

$$\det(A) = 1\cdot(-1)^{1+3}\det(A_{1,3}) + 0\cdot\text{whatever} + (-2)(-1)^{3+3}\det(A_{3,3})$$

$$= \det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} + 0 + (-2)\det\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= 3 + (-2)(-1) = 5 \checkmark$$

Example

$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -2 \end{pmatrix}$. Calculate $\det(A)$ ~~3~~ 2 different ways.

Expand across row 2.

$$\det(A) = (1)(-1)^{2+1}\det(A_{2,1}) + (-1)(-1)^{2+2}\det(A_{2,2}) + 0\cdot\text{whatever}$$

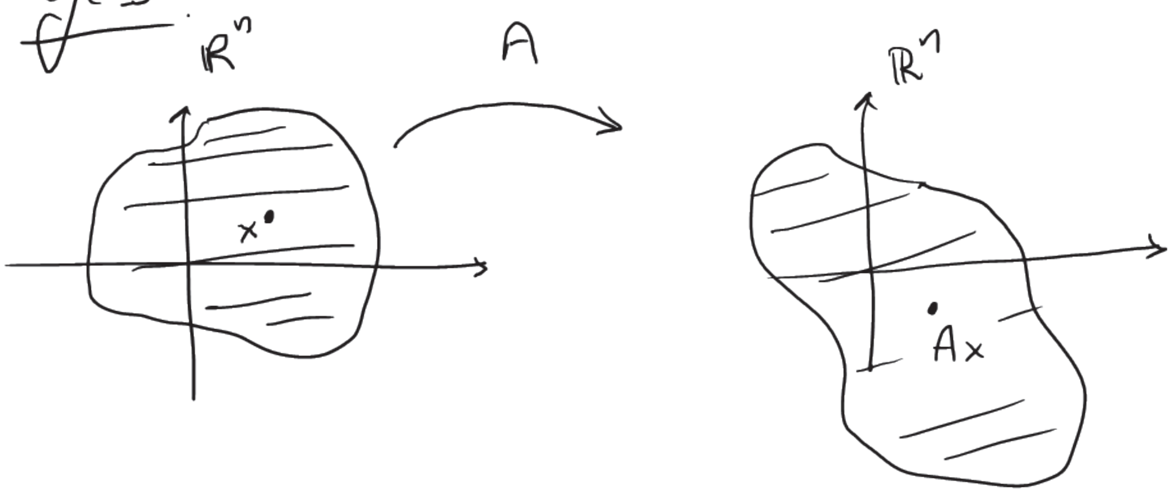
$$= (-1)\det\begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} + (-1)\det\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} + 0$$

$$= (-1)(-6) + (-1)(1) = 5 \checkmark$$

Questions (cont.)

Is there a geometric interpretation of the determinant of an nxn matrix?

yes



n dimensional volume of this new shape is equal to $|\det(A)|$ times the n dimensional volume of the original shape.

Question

How do elementary row operations effect the determinant of a matrix?

$R_i \leftrightarrow R_j$ $\quad i \neq j$ $\quad \equiv$ multiplies by (-1)

$\alpha R_i \to R_i$ $\quad \alpha \neq 0$ $\quad \equiv$ multiplies by α

$\alpha R_i + R_j \to R_j$ $\quad i \neq j$ $\quad \equiv$ no change

This is how determinants are computed by computers.

Do elementary row operations until all entries below the diagonal are zero. Keep track of the row operations, and use this information along with the product of the entries on the diagonal of the final matrix to get the determinant.

Why? Formula on $n \times n$ matrix $\Rightarrow$ $n!$ Calculations

Row ops on $n \times n$ matrix $\Rightarrow$ $\approx \frac{n^3}{3}$ operations

$n = 20$ $\Rightarrow$ $n! \equiv$ Huge. You'll die before this is done

$\frac{n^3}{3} \equiv$ not huge. This takes a fraction of a second.

Question

How is the determinant of a matrix related to the determinant of the transpose of the matrix? Of the inverse of the matrix (if it exists)?

transpose $\longleftrightarrow$ switch rows + columns.

Transpose of $A \equiv A^T$

$$\det(A) = \det(A^T)$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

This might hint to the fact that a matrix A is invertible if and only if its determinant is nonzero.

Question

Suppose A and B are nxn matrices and α is a scalar. Determine which of the following are true:

- det(A + B) = det(A) + det(B) Not unless A is 1x1.
- det(A B) = det(A) det(B) Yes. Always.
- det(α A) = α det(A) No. $\alpha^n \det(A) = \det(\alpha A)$.

only helps if the vectors form a square matrix.

Questions (cont.)

How is the idea of determinant related to linear independence?

If A is $n \times n$ then the columns of A are linearly independent if and only if $\det(A) \neq 0$.

Definition

An nxn matrix A is **nonsingular** if and only if

$\det(A) \neq 0$

Equivalently the columns of A are Lin. ind.

Equivalently the only sol'n to $Ax = \vec{0}$ is the trivial one.
i.e. $x = \vec{0}$.

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2. Give the determinant of $\begin{pmatrix} \boxed{-1 \quad 0 \quad 1} \\ 0 \quad 1 \quad 1 \\ 1 \quad -2 \quad 0 \end{pmatrix}$.

a. -2

b. -3

c. 4

d. None of these

Expand across row 1.

$= (-1)(-1)^{1+1}\det\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} + 0 + 1\cdot(-1)^{1+3}\det\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

$= (-1)(2) + (-1) = -3.$

3. A 4. A. 5. C

$A_{n \times n}$

Eigenvalues and Eigenvectors

Definition – Part 1

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in R$ is a real eigenvalue of A if and only if there is a nonzero vector $x \in R^n$ so that $\boxed{Ax = \lambda x}$. In this case, the vector x is referred to as an eigenvector associated with the real eigenvalue λ.

real matrix

Definition – Part 2

complex #5

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in C$ with $im(\lambda) \neq 0$ is a complex eigenvalue of A if and only if there is a nonzero vector $x \in C^n$ so that $Ax = \lambda x$. In this case, the vector x is referred to as an eigenvector associated with the complex eigenvalue λ.

complex vector

Hm... How can I find values λ so that we get nonzero vectors x so that

$$Ax = \lambda x \quad ?$$

Rewrite

$$Ax - \lambda x = \vec{0}$$

$$Ax - \lambda I_n x = \vec{0}$$

$$\underbrace{(A - \lambda I_n)}_{\text{matrix}} x = \vec{0} \quad \leftarrow \text{homogeneous equation.}$$

This has a nontrivial sol'n x iff

$$\det(A - \lambda I_n) = 0.$$

characteristic polynomial for A.

So, to find the eigenvalues of an nxn matrix A, we first find the characterstic polynomial given by det($A - \lambda I$).

Then we find the roots of this polynomial. These are the eigenvalues. Once we have these, we look for the nonzero solutions to

$(A - \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\lambda} I)x = 0$. These are the associated eigen vectors.

Question: How do we find the eigenvalues and associated eigenvectors of a real square matrix A?

Definition: Characteristic Polynomial

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} 7 & 6 \\ -9 & -8 \end{pmatrix}$.

A - λI

$$\det(A - \lambda I) = \det\begin{pmatrix} 7-\lambda & 6 \\ -9 & -8-\lambda \end{pmatrix}$$

$$= (7-\lambda)(-8-\lambda) + 54 = -56 + \lambda + \lambda^2 + 54$$

$$= \lambda^2 + \lambda - 2.$$ ← characteristic polynomial

Solve $\lambda^2 + \lambda - 2 = 0 \iff (\lambda+2)(\lambda-1) = 0$

$\iff \lambda = -2, \lambda = 1.$

i.e. the eigenvalues are -2 and 1.

let's get the associated eigenvectors.

$\lambda = -2$: Find the nontrivial sol'ns to

$(A - (-2)I)x = \vec{0}$. i.e. $\begin{pmatrix} 9 & 6 \\ -9 & -6 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$9x_1 + 6x_2 = 0 \Rightarrow x_1 = -\frac{2}{3}x_2$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3\, x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} x_2, \quad x_2 \neq 0$

$\therefore$ the eigenvectors assoc. with the eigenvalue $\lambda = -2$ are nonzero scalar multiples of $\begin{pmatrix} -2/3 \\ 1 \end{pmatrix}$.

$\underline{\lambda = 1}$: Find the nontrivial sol'ns to

$$(A - (1)I)x = \vec{0}$$

i.e. $\begin{pmatrix} 6 & 6 \\ -9 & -9 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$6x_1 + 6x_2 = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2 \quad \text{for } x_2 \neq 0.$$

$\therefore$ the eigenvectors assoc. with the eigenvalue 1 are all nonzero scalar multiples of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$ ← characteristic polynomial of A

Solve $\lambda^2 + 1 = 0 \iff \lambda = \pm i$. ← conjugate pair b/c A is real

$\underline{\lambda = i}$: Find nontrivial sol'ns to

$(A - iI)x = \vec{0}$. i.e. $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-ix_1 - x_2 = 0$ Note: Mult. First Eq. by i to get the second.

equivalent

$x_1 - ix_2 = 0$

$\hookrightarrow x_1 = ix_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} x_2$

$x_2 \neq 0$

$\therefore$ The eigenvectors assoc with $\lambda = i$ are nonzero multiples of $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

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complex nonzero.

Warning:

ex. $(2-i)\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2i \\ 2-i \end{pmatrix}$ is an eigen vector assoc. with $\lambda = i$.

Trick:

For $\lambda = -i$: Just conjugate $\begin{pmatrix} i \\ 1 \end{pmatrix}$. This gives $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ AND the eigenvectors assoc. with $\lambda = -i$ will be nonzero scalar multiples of $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Aside. $a, b \in \mathbb{R}$.

$\underbrace{\text{conjugate of } a+bi}_{=\ \overline{a+bi}} = a - bi$

$\overline{a+bi} = a - bi$

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -3 & 2 & -2 \end{pmatrix}$.