Recall: $\boxed{\text{Linear Systems of Equations}}$

Solution Process Via Elementary Row Operations

$$A x = b \longleftrightarrow$$

This system can have 0, 1 or infinitely many sol'ns.

A: $m \times n$ known matrix

x: unknown $\underline{\underline{n \times 1}}$

b: known $m \times 1$ matrix } vector

Method:

1. Form the augmented matrix $\begin{pmatrix} A & b \end{pmatrix}$ $m \times (n+1)$
2. Use Elem row operations to solve.
   Goal: work Top to bottom, left to right to clear out entries below diagonal
3. Write the solution(s), if they exist.

Homogeneous vs Nonhomogeneous Linear Systems

$A x = \vec{0}$ ← column of all 0 entries

Note: $x = \vec{0}$ is always a sol'n.

$A x = b$ with $b \neq \vec{0}$

Note: $\begin{pmatrix} A & \vec{0} \end{pmatrix}$ vs $\begin{pmatrix} A & b \end{pmatrix}$

Same Elem Row Operations

↓ sol'n. Always solvable

↓ sol'n. might or might not have a sol'n.

Note: If Ax=0 has only x=0 as a solution, then Ax=b can have NO MORE than one solution!!

Row Reduced Echelon Form

$\boxed{\text{(RREF)}}$

Motivating Example:

$$2x_1 + 3x_2 + 3x_3 - x_4 = 2$$
$$x_1 - 2x_2 + 4x_3 + x_4 = -1$$
$$-x_1 + 3x_3 + 2x_4 = 1$$

1. Form the augmented matrix
2. Use Elem. Row Ops. to obtain

$$\begin{pmatrix} 1, & -2, & 4, & 1, & -1; \\ 0, & 1, & -5/7, & -3/7, & 4/7; \\ 0, & 0, & 39/7, & 15/7, & 8/7; \end{pmatrix}$$

$\rightarrow x_2 - \frac{5}{7}x_3 - \frac{3}{7}x_4 = \frac{4}{7}$

$\rightarrow \frac{39}{7}x_3 + \frac{15}{7}x_4 = \frac{8}{7}$

$\boxed{x_3 = -\frac{15}{39}x_4 + \frac{8}{39}}$

Plug in and solve for x_2

Then use row 1, x_2 and x_3 to solve for x_1.

Algebraic Pain!!

A (possibly) better way... Do a few more elementary row operations.

$$\begin{pmatrix} 1, & -2, & 4, & 1, & -1; \\ 0, & 1, & -5/7, & -3/7, & 4/7; \\ 0, & 0, & 39/7, & 15/7, & 8/7; \end{pmatrix}$$

Convert to 0

Convert to 1

(7/39)R3 -> R3 gives

$$\begin{pmatrix} 1, & -2, & 4, & 1, & -1; \\ 0, & 1, & -5/7, & -3/7, & 4/7; \\ 0, & 0, & 1, & 5/13, & 8/39; \end{pmatrix}$$

(2)R2 + (1)R1 -> R1 gives

$$\begin{pmatrix} 1, & 0, & 18/7, & 1/7, & 1/7; \\ 0, & 1, & -5/7, & -3/7, & 4/7; \\ 0, & 0, & 1, & 5/13, & 8/39; \end{pmatrix}$$

(5/7)R3 + (1)R2 -> R2 gives

$$\begin{pmatrix} 1, & 0, & 18/7, & 1/7, & 1/7; \\ 0, & 1, & 0, & -2/13, & 28/39; \\ 0, & 0, & 1, & 5/13, & 8/39; \end{pmatrix}$$

(-18/7)R3 + (1)R1 -> R1 gives

$$\begin{pmatrix} 1, & 0, & 0, & -11/13, & -5/13; \\ 0, & 1, & 0, & -2/13, & 28/39; \\ \underbrace{0, & 0, & 1, & 5/13, & 8/39;}_{x_3 + \frac{5}{13}x_4 = \frac{8}{39}} \end{pmatrix}$$

$\rightarrow x_1 = \frac{11}{13}x_4 - \frac{5}{13}$

$\rightarrow x_2 = \frac{2}{13}x_4 + \frac{28}{39}$

$\rightarrow x_3 = -\frac{5}{13}x_4 + \frac{8}{39}$

$x_4 =$ anything.

Note that doing the elementary row operations in the manner than we did allowed us to simply write the solution down.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{11}{13}x_4 - \frac{15}{13} \\ \frac{2}{13}x_4 + \frac{28}{39} \\ -\frac{5}{13}x_4 + \frac{8}{39} \\ x_4 \end{pmatrix}$$

Infinitely many sol'ns. One for each choice of x_4.

I used the Matrix Calculator linked from http://online.math.uh.edu

The augmented matrix is
2, 3, 3, -1, 2;
1, -2, 4, 1, -1;
-1, 0, 3, 2, 1;
R1 <-> R2 gives
1, -2, 4, 1, -1;
2, 3, 3, -1, 2;
-1, 0, 3, 2, 1;
(-2)R1 + (1)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
-1, 0, 3, 2, 1;
(1)R1 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 7, -5, -3, 4;
0, -2, 7, 3, 0;
(1/7)R2 -> R2 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, -2, 7, 3, 0;
(2)R2 + (1)R3 -> R3 gives
1, -2, 4, 1, -1;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;

(2)R2 + (1)R1 -> R1 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 39/7, 15/7, 8/7;
(7/39)R3 -> R3 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, -5/7, -3/7, 4/7;
0, 0, 1, 5/13, 8/39;
(5/7)R3 + (1)R2 -> R2 gives
1, 0, 18/7, 1/7, 1/7;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;
(-18/7)R3 + (1)R1 -> R1 gives
1, 0, 0, -11/13, -5/13;
0, 1, 0, -2/13, 28/39;
0, 0, 1, 5/13, 8/39;

Row Reduced Echelon Form (RREF)

General Process: Apply elementary row operations to a matrix until

1. The first nonzero entry in each row (if there is one) is 1. (These are called leading entries.)
2. The leading entry in a row is to the right of a leading entry in a row above.
3. The entries above and below leading entries are all 0.
4. Rows of all zeros should be at the bottom.

Examples: Which of the following augmented matrices are in RREF?

$$\begin{pmatrix} 1 & 3 & 3 & -1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ?$$

NO

should be 0.

leading entries

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} ?$$

NO

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ?$$

Yes ✓

Example:

The RREF of the augmented matrix for the linear system

$$-x_1 + 3x_2 + x_3 - x_4 = -4$$
$$-3x_1 + 8x_2 + 4x_3 - 3x_4 = -14$$
$$4x_1 - 11x_2 - 6x_3 + 7x_4 = 20$$

is

$$\begin{pmatrix} 1 & 0 & 0 & -11 & 2 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -3 & -2 \end{pmatrix}$$

Solve the system.

$$\left.\begin{aligned} x_1 &= 11x_4 + 2 \\ x_2 &= 3x_4 \\ x_3 &= 3x_4 - 2 \\ x_4 &= \text{anything} \end{aligned}\right\}$$

Infinitely many sol'ns.

Computing the Inverse of an Invertible Matrix Using RREF

1. Form $\begin{pmatrix} A & I_n \end{pmatrix}$

2. Push to RREF

$$\begin{pmatrix} \boxed{?} & B \end{pmatrix}$$

If this is I_n then $B = A^{-1}$

If this is not I_n then A is not invertible.

Question

Let $A \in R^{m,n}$ and $x \in R^n$. Denote the columns of A by $\vec{c}_1, \dots, \vec{c}_n$ and suppose $x = (x_i)$. What is Ax?

$$A = \begin{pmatrix} \vec{C}_1 & \vec{C}_2 & \cdots & \vec{C}_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Ax = x_1 \vec{C}_1 + x_2 \vec{C}_2 + \cdots + x_n \vec{C}_n \equiv$$ a linear combination of the columns in A

Key phrase: linear combination.

$$\begin{array}{ccccccc} \vec{C}_1 & \vec{C}_2 & & & & x_1 \cdot \vec{C}_1 & + \; x_2 \cdot \vec{C}_2 \end{array}$$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -4 \\ 5 \end{pmatrix}}_{x} = \begin{pmatrix} 2 \cdot (-4) + 1 \cdot (5) \\ -1 \cdot (-4) + 3 \cdot (5) \end{pmatrix} = -4 \boxed{\begin{pmatrix} 2 \\ -1 \end{pmatrix}} + 5 \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

$$A = \begin{pmatrix} \vec{C}_1 & \vec{C}_2 & \cdots & \vec{C}_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Important Note: You can only solve $Ax = b$ if b is a linear combination of the columns of A.

Why?

$$Ax = b$$

$$\underbrace{x_1 \vec{C}_1 + x_2 \vec{C}_2 + \cdots + x_n \vec{C}_n}_{\text{linear combination of the columns of A}} = b$$

Note: $Ax = \vec{0}$ ← Always has at least one sol'n. Namely $x = \vec{0}$.

$$x_1 \vec{C}_1 + x_2 \vec{C}_2 + \cdots + x_n \vec{C}_n = \vec{0}$$

All $x_i = 0$ makes this the trivial linear combination of the columns.

Note: If there is a nonzero solution to Ax=0, then we have a nontrivial linear combination of the columns that gives the 0 vector.

Definition

Let $\vec{c}_1, \dots, \vec{c}_n \in R^m$. The set $\{\vec{c}_1, \dots, \vec{c}_n\}$ is linearly independent if and only if $x_1 \vec{c}_1 + \dots + x_n \vec{c}_n = \vec{0}$, with $x_i \in R$, implies $x_i = 0$ for all $i = 1, \dots, n$. Otherwise, we say the set is linearly dependent.

i.e. $\{\vec{C}_1, \dots, \vec{C}_n\}$ is linearly independent iff the only linear combination that gives $\vec{0}$ is the trivial one.

Question

How is linear independence related to solving $Ax = 0$?

Start with $\{\vec{c}_1, \ldots, \vec{c}_n\}$. ① Form

$$A = \left(\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_n\right)$$

② Solve $A x = \vec{0}$.

If the only sol'n is $x = \vec{0}$ then $\{\vec{c}_1, \ldots, \vec{c}_n\}$ is linearly independent. Otherwise, it is linearly dependent.

Determine whether the set $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

$\vec{c}_1$ $\vec{c}_2$ $\vec{c}_3$

Linearly dependent.

① Form $A = \begin{pmatrix} -1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

② Solve $A x = \vec{0}$

the augmented matrix is
-1, -2, 1, 0;
2, 1, 1, 0;
1, 0, 1, 0;
The rref is
1, 0, 1, 0;
0, 1, -1, 0;
0, 0, 0, 0;

$\Rightarrow$ $x_1 = -x_3$, $x_2 = x_3$, $x_3 = \text{anything}$

Infinitely many sol'ns. $\therefore$ $x = \vec{0}$ IS NOT the only sol'n.

Note: $x_3 = 1 \Rightarrow (-1)\vec{c}_1 + (1)\vec{c}_2 + (1)\vec{c}_3 = \vec{0}$

Determine the value(s) of c so that $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2c \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -1 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

① $A = \begin{pmatrix} -1 & -2c & c \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

② Solve $A x = \vec{0}$.

$$\begin{pmatrix} -1 & -2c & c & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ \boxed{2} & 1 & -1 & 0 \\ \boxed{-1} & \boxed{-2c} & c & 0 \end{pmatrix}$

$-2R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \boxed{-2c} & 1+c & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \boxed{-2c} & 1+c & 0 \end{pmatrix}$$

$2c R_2 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & \boxed{1-5c} & 0 \end{pmatrix}$$

For linear independence, I need for the only solution that comes from here to be $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

This all hinges on $1-5c$.

The vectors are linearly independent iff $c \neq \frac{1}{5}$.

If $1 - 5c \neq 0$ then we get $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Otherwise, there will be a nontrivial sol'n.

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1. Determine whether the set $\left\{\begin{pmatrix}-1\\-2\\1\end{pmatrix},\begin{pmatrix}-2\\-1\\0\end{pmatrix},\begin{pmatrix}1\\-3\\1\end{pmatrix}\right\}$

is linearly independent or linearly dependent.

a. linearly independent

b. linearly dependent

c. there is not enough information

$A=\begin{pmatrix}-1&-2&1\\-2&-1&-3\\1&0&1\end{pmatrix}$

Solve $Ax=\vec{0}$

$$\begin{pmatrix}-1&-2&1&0\\ \boxed{-2}&-1&-3&0\\ \boxed{1}&0&1&0\end{pmatrix}$$

$-2R_1+R_2\to R_2$
$R_1+R_3\to R_3$

$$\begin{pmatrix}-1&-2&1&0\\0&3&-5&0\\0&-2&2&0\end{pmatrix}$$

$\frac{2}{3}R_2+R_3\to R_3$

$$\begin{pmatrix}-1&-2&1&0\\0&3&-5&0\\0&0&-\frac{4}{3}&0\end{pmatrix}\Rightarrow$$

only sol'n is $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\therefore$ The vectors are linearly independent.

ex. $A=(3)\Rightarrow \det(A)=3$

ex. $A=\begin{pmatrix}2&-1\\3&2\end{pmatrix}\Rightarrow \det(A)=7$

Determinants

How do we take the determinant of a 1x1 matrix?

$A=(\alpha)$ $\quad\det(A)=\alpha$

How do we take the determinant of a 2x2 matrix?

$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $\quad\det(A)=ad-bc$

How do we take the determinant of an nxn matrix?

Much more complicated.

We need some notation.

① Start: A is an $n\times n$ matrix.

Spse $A=\begin{pmatrix}a_{1,1}&a_{1,2}&\cdots&a_{1,n}\\ \vdots&&&\vdots\\ a_{n,1}&a_{n,2}&\cdots&a_{n,n}\end{pmatrix}$

② $A_{i,j}$ is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column from A.

Ex. $A=\begin{pmatrix}1&2&-1\\1&1&3\\2&-6&5\end{pmatrix}\Rightarrow A_{2,3}=\begin{pmatrix}1&2\\2&-6\end{pmatrix}$

$A_{3,2}=\begin{pmatrix}1&-1\\1&3\end{pmatrix}$

Spse A is $n\times n$ as above.

Each of the following formulas will give the same value. These are all called the determinant of A.

1. Expanding across the k^{th} row.

$$\det(A)=\sum_{j=1}^{n} a_{k,j}(-1)^{k+j}\det(A_{k,j})$$

2. Expanding down the k^{th} column.

$$\det(A)=\sum_{i=1}^{n} a_{i,k}(-1)^{i+k}\det(A_{i,k})$$

The decision to expand across a particular row or down a particular column can save you some time (if you choose wisely).

Expand down the 2nd column ← smart approach

$\det(A) = 0 + 3(-1)^{2+2}\det(A_{2,2}) + 0$

$= 3\det\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$

$= 3\left(8 - (-3)\cdot 1\right) = \underline{\underline{33}}$

Lots of 0 entries.

Compute the determinant of $A = \begin{pmatrix} 2 & \boxed{0} & 1 \\ -1 & 3 & 2 \\ -3 & 0 & 4 \end{pmatrix}$ in at least 2 different ways.

Expand across 2nd row. ← not so smart

$\det(A) = (-1)\cdot(-1)^{2+1}\det(A_{2,1}) + 3(-1)^{2+2}\det(A_{2,2}) + 2(-1)^{2+3}\det(A_{2,3})$

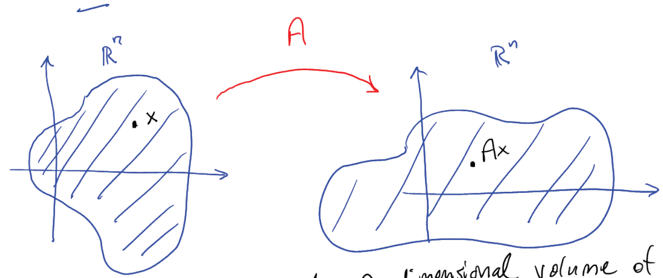
$= \det\begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} + 3\det\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix} + (-2)\det\begin{pmatrix} 2 & 0 \\ -3 & 0 \end{pmatrix}$

$= 0 + 3\left(8 - (3)\cdot 1\right) + 0 = \underline{\underline{33}}$

Questions (cont.)

Is there a geometric interpretation of the determinant of an nxn matrix?

Yes



the n-dimensional volume of this is given by $|\det(A)|$ times the n-dimensional volume of the original region.

Question

How do elementary row operations effect the determinant of a matrix?

Action		Effect on determinant
$R_i \to R_j$	$i \neq j$	mult by (-1)
$\alpha R_i \to R_i$	$\alpha \neq 0$	mult α
$\alpha R_i + R_j \to R_j$	$i \neq j$	No change.

So, if we use elementary row operations until the matrix has all zeros below the diagonal, then we can get the determinate by keeping track of the changes, and then multiplying by the product of the entries in the last matrix.

Note: 20! is huge.

You might die before your computer does 20! calculations.

Why bother?????

Formula based determinant requires $\approx n!$ calculations

Organized row reduction method to find the determinant $\approx \frac{n^3}{3}$ Calculations

$n = 20 \quad \frac{20^3}{3} = \frac{8000}{3}$ Easy

Question

How is the determinant of a matrix related to the determinant of the transpose of the matrix? Of the inverse of the matrix (if it exists)?

To get the transpose, switch rows and columns.

Notation: transpose of $A = A^T$

$\det(A) = \det(A^T)$.

$\det(A^{-1}) = \frac{1}{\det(A)}$

A matrix is invertible if and only if det(A) is nonzero.

Question

Suppose A and B are nxn matrices and α is a scalar. Determine which of the following are true:

- det(A + B) = det(A) + det(B) False
- det(A B) = det(A) det(B) True
- det(α A) = α det(A) False

$\det(\alpha A) = \alpha^n \det(A)$.

Questions (cont.)

How is the idea of determinant related to linear independence?

n vectors of length n are Lin. Ind. iff the determinant of matrix of these vectors is NOT ZERO.

Definition

An nxn matrix *A* is **nonsingular** if and only if

$\det(A) \neq 0$

Equivalently

A^{-1} exists

Equivalently

$Ax = 0$ only has $x = 0$ as a sol'n.

Equivalently

Columns of A are Lin. Ind.

3. A

4. C

5. B

EMCF06b

2. Give the determinant of $\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 2 \\ -3 & 0 & -2 \end{pmatrix}$.

a. -5

b. -7

c. 6

d. None of these

$= (-3)(-1)^{3+1} \det(A_{3,1}) + 0 + (-2)(-1)^{3+3} \det(A_{3,3})$

$= -3 \det\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} - 2 \det\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$= -3(2 - -1) - 2(-2 - -1) = -9 + 2 = -7$

$n \times n$

Eigenvalues and Eigenvectors

Definition – Part 1

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in R$ is a real eigenvalue of A if and only if there is a nonzero vector $x \in R^n$ so that $\boxed{Ax = \lambda x}$. In this case, the vector x is referred to as an eigenvector associated with the real eigenvalue λ.

Definition – Part 2

Suppose $A \in R^{n \times n}$. We say that a number $\lambda \in C$ with $im(\lambda) \neq 0$ is a complex eigenvalue of A if and only if there is a nonzero vector $x \in C^n$ so that $Ax = \lambda x$. In this case, the vector x is referred to as an eigenvector associated with the complex eigenvalue λ.

Question: How do we find the eigenvalues and associated eigenvectors of a real square matrix A?

We need nonzero vectors x and scalars λ so that

$$Ax = \lambda x$$

rewrite

$$Ax - \lambda x = \vec{0}$$

$$Ax - \lambda I_n x = \vec{0}$$

$$\underbrace{(A - \lambda I_n)}_{n \times n \text{ matrix}} x = \vec{0}$$

Definition: Characteristic Polynomial

Need $\det(A - \lambda I_n) = 0$

Characteristic polynomial of A.

So, to find the eigenvalues of A, you solve $\det(A - \lambda I) = 0$

Then for each eig. value λ, find the nonzero vectors x solving $(A - \lambda I)x = \vec{0}$ to get the associated eig. vectors.

Example

Find the eigenvalues and associated eigenvectors for the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

1. Form the characteristic polynomial.

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$$

2. Solve $\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \underbrace{\lambda = 3, \ \lambda = 1}_{\text{Eigenvalues}}$

Let's get the associated eigenvectors.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ Eigenvalues $\lambda = 3$, $\lambda = 1$.

$\lambda = 3$: Find the nonzero sol'ns to

$$(A - 3I)x = \vec{0}$$

i.e. $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow x_1 = x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2, \quad x_2 \neq 0$$

Therefore, the eigen vectors associated with the eigenvalue 3 are the nonzero scalar multiples of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

representative eigenvector.

Note: Infinitely many eig vectors.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda = 1$: Find the nonzero sol'ns to

$$(A - 1 \cdot I)x = \vec{0}$$

i.e. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow x_1 = -x_2$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2, \quad x_2 \neq 0$$

The eigen vectors associated with the eigen value 1 are the nonzero scalar multiples of

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

Example

Find the eigenvalues and associated eigenvectors

for the matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -3 & 2 & -2 \end{pmatrix}$.