Notes:

- See the discussion board for curve information and instructions for obtaining a scanned graded copy of your midterm exam.
- Homework is posted.
- **There is no excuse for not having excellent online quiz grades!**

Open EMCF07a.

One further eigenvalue/eigenvector example.

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

1. Find eigenvalues.

<u>By hand</u> – Find characteristic polynomial.

$$\det(A-\lambda I) = \det\begin{pmatrix} \boxed{2-\lambda} & 2 & 3 \\ \boxed{1} & 2-\lambda & 1 \\ \boxed{2} & -2 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(-1)^{1+1}\det\begin{pmatrix} 2-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} + 1\cdot(-1)^{2+1}\det\begin{pmatrix} 2 & 3 \\ -2 & 1-\lambda \end{pmatrix}$$

$$+ 2(-1)^{3+1}\det\begin{pmatrix} 2 & 3 \\ 2-\lambda & 1 \end{pmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(1-\lambda)+2\right] - \left[2(1-\lambda)+6\right] + 2\left[2-3(2-\lambda)\right]$$

$$= (2-\lambda)\left[2-3\lambda+\lambda^2+2\right] - \left[2-2\lambda+6\right] + 2\left[2-6+3\lambda\right]$$

$$= (2-\lambda)\left[4-3\lambda+\lambda^2\right] - \left[8-2\lambda\right] + 2\left[-4+3\lambda\right]$$

$= 8 - 10\lambda + 5\lambda^2 - \lambda^3 - 8 + 2\lambda - 8 + 6\lambda$

$= -8 - 2\lambda + 5\lambda^2 - \lambda^3$

Now we find the roots. i.e. Solve

$$-8 - 2\lambda + 5\lambda^2 - \lambda^3 = 0$$

$$\boxed{\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0}$$

$\lambda = -1$ works $\quad (-1)^3 - 5(-1)^2 + 2(-1) + 8 = 0$ ✓

$\Rightarrow$ $\lambda + 1$ is a factor

Division:

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 8 \\ \hline \lambda + 1 & \lambda^3 - 5\lambda^2 + 2\lambda + 8 \\ & -(\lambda^3 + \lambda^2) \\ \hline & -6\lambda^2 + 2\lambda + 8 \\ & -(-6\lambda^2 - 6\lambda) \\ \hline & 8\lambda + 8 \\ & -(8\lambda + 8) \\ \hline & 0 \end{array}$$

$$(\lambda+1)(\lambda^2 - 6\lambda + 8) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda-4) = 0$$

$\therefore$ The eigen values are $\lambda = -1, \lambda = 2, \lambda = 4$.

Let's get the assoc. eigen vectors.

$\lambda = -1$: Find the nonzero vectors solving

$$(A - (-1)I)x = \vec{0}$$

Note: $A-(-1)I = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{pmatrix}$

Solve $\begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{pmatrix} x = \vec{0}$

Just the nonzero sol'ns.

$$\left(\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ \boxed{3} & 2 & 3 & 0 \\ \boxed{2} & -2 & 2 & 0 \end{array}\right)$$

$-3R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & \boxed{-8} & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = 0$

$x_1 = -x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad x_3 \neq 0.$$

i.e. The eigenvectors assoc. with $\lambda = -1$ are the nonzero multiples of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda = 4$: the augmented matrix for $(A - 4I)x = 0$ is

-2, 2, 3, 0;
1, -2, 1, 0;
2, -2, -3, 0;

The rref is

1, 0, -4, 0; $\rightarrow$ $x_1 = 4x_3$
0, 1, -5/2, 0; $\rightarrow$ $x_2 = \frac{5}{2}x_3$
0, 0, 0, 0;

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_3 \\ \frac{5}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix}, \quad x_3 \neq 0$$

$\therefore$ the eigenvectors assoc. with $\lambda = 4$ are nonzero multiples of $\begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

the augmented matrix associated with (A-2I)x=0 is

0, 2, 3, 0;
1, 0, 1, 0;
2, -2, -1, 0;

The rref is

1, 0, 1, 0; → $x_1 = -x_3$
0, 1, 3/2, 0; → $x_2 = -\frac{3}{2}x_3$
0, 0, 0, 0;

$1 = -x_3$

$x_3 = -1.$

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1. One eigenvector associated with the eigenvalue 2 has 1 as its first entry. What is its third entry?

a. 2
b. -2
c. -1
d. 0
e. None of these.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -3/2\, x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix}, \quad x_3 \neq 0$$

∴ the eigenvectors assoc. with $\lambda = 2$ are nonzero multiples of $\begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix}$.

Solutions to linear first order systems.

Motivating Example: Solve

System of differential equations.

$$\begin{cases} x' = x - 3y \\ y' = -2x + 2y \end{cases}$$

Here $x = x(t)$, $y = y(t)$

$$x'(t) = x(t) - 3y(t)$$

$$y'(t) = -2x(t) + 2y(t).$$

writing the "of t's" doesn't help us solve the system.

Using matrix vector form does help.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x - 3y \\ -2x + 2y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So, if we name $\boxed{u = \begin{pmatrix} x \\ y \end{pmatrix}}$, we have

$u' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} u$. i.e. $u' = Au$ with $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$

If we just walked in, we might think $A \equiv$ constant. If so, then

$$u' = Au \quad \text{leads to} \quad u = c_0 e^{At}.$$

BUT A is a matrix. Hm... It's still tempting.

Let's pretend we don't know how we got this...

Q: Are there sol'ns that look like

$$\underbrace{\text{something}}_{\text{vector!}} \cdot \underbrace{e^{\text{constant} \cdot t}}_{\text{scalar}} \; ?$$

A: $\longrightarrow$

$$\vec{v} e^{kt}.$$ Let's try it.

Note: If $\vec{v} = \vec{0}$ then $\vec{v}e^{kt} = \vec{0}$. Nothing to do.

Subst. this into

$$u' = Au$$

Spse $\vec{v} \neq \vec{0}$

$$(\vec{v} e^{kt})' = A(\vec{v} e^{kt})$$

$$\underbrace{k\vec{v}}\,\cancel{e^{kt}} = \underbrace{A\vec{v}}\,\cancel{e^{kt}}$$

Look! $A\vec{v} = k\vec{v}$. i.e. $\vec{v}$ is an eigenvector assoc. with the eigenvalue k.

Back to $x' = x - 3y$

$y' = -2x + 2y$

let's get the eigen value / eigen vector pairs for $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$

1, -3;
-2, 2;
The characteristic polynomial is -4 - (3) z + z^2
The eigenvalues of A are the roots. i.e. -1 and 4.
Let's find the eigen vectors associated with -1.

The rref of the augmented matrix associated with (A-(-1)I)x=0 is
1, -3/2, 0;
0, 0, 0;

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2\, x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} x_2, \quad x_2 \neq 0$ 1

$\therefore$ the eig. vects assoc with $\lambda = -1$ are nonzero mult of $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

The rref of the augmented matrix associated with (A-4I)x=0 is
1, 1, 0;
0, 0, 0;

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2, \quad x_2 \neq 0$ 2

$\therefore$ the eig. vects assoc with $\lambda = 4$ are nonzero mults of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So $c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t}$ is a sol'n for any constant c_1

AND $c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$ is a sol'n for any constant c_2.

Q: How do we get the general sol'n to

$u' = Au$ with $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$?

A: use linearity to join these sol'ns.

Note: If $u(t) = c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$

then $u'(t) = -c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + 4c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$

$\boxed{\text{recall: } Ac_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = -c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \text{ and } Ac_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 4c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$$= Ac_1 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{-t} + Ac_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

$$= A\left(c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} \right)$$

$$= A\,u(t)$$

i.e. $u(t)$ solves. ALSO, $u(t)$ is the general sol'n.

why? This allows us to "cover" any initial data.

Huh?

$$\begin{cases} x' = x - 3y \\ y' = -2x + 2y \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

$u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\left.\begin{matrix} x(0) = x_0 \\ y(0) = y_0 \end{matrix}\right\}$ initial data. Here x_0, y_0 are known.

$$u(t) = c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

Need $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

i.e. Given $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, we need to assured that we can find c_1, c_2 so that this holds.

$$\underbrace{\begin{pmatrix} 3/2 & -1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

known.

Can we solve for c_1 and c_2?

YES. we can solve uniquely b/c

$\begin{pmatrix} 3/2 & -1 \\ 1 & 1 \end{pmatrix}$ is nonsingular.

(i.e. the columns are L.I.)

(i.e. determinant $\neq 0$)

(i.e. $\begin{pmatrix} 3/2 & -1 \\ 1 & 1 \end{pmatrix}$ is invertible)

Initial Value Problem

$$x' = x - 3y$$
$$y' = -2x + 2y$$

Same system of ODEs as above

initial data $\begin{cases} x(0) = -2 \\ y(0) = 7 \end{cases}$

The general sol'n to the system of ODEs is

$$u(t) = c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

To satisfy the initial data, we need the sol'n at $t=0$ to be $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$.

i.e.

$$c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

Solve: $c_1 = 2$ $c_2 = 5$ do it!

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2\begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + 5\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

i.e.,
$$\boxed{\begin{aligned} x(t) &= 3e^{-t} - 5e^{4t} \\ y(t) &= 2e^{-t} + 5e^{4t} \end{aligned}}$$

5 min.

In General...

How do we solve $u' = Au$ when A is a real nxn matrix with n distinct real eigenvalues?

Setting: $\lambda_1, \lambda_2, \dots, \lambda_n$ distinct real eigen values

$\underbrace{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n}_{\text{these will be linearly independent.}}$ representative eigen vectors.

Term: Fundamental Matrix

$$\Rightarrow u(t) = c_1 \vec{u}_1 e^{\lambda_1 t} + c_2 \vec{u}_2 e^{\lambda_2 t} + \cdots + c_n \vec{u}_n e^{\lambda_n t}$$

general sol'n.

arbitrary constants

AND
$$= \begin{pmatrix} \vec{u}_1 e^{\lambda_1 t} & \vec{u}_2 e^{\lambda_2 t} & \cdots & \vec{u}_n e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

each column is a sol'n, and they are L.I.

$u' = Au$

A fundamental matrix assoc with this system is a matrix function

$U(t)$ which is $n \times n$

and satisfies:

1. Each column is a sol'n to
$$u' = Au$$
2. $U(t)$ is nonsingular for each t.
i.e. $\det U(t) \neq 0$ for all t.

Note: If $U(t)$ is a fundamental matrix, then the general sol'n to $u' = Au$ is given by

$$U(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

If $u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ the system is

$$x' = 2x + 2y + 3z$$
$$y' = x + 2y + z$$
$$z' = 2x - 2y + z$$

Example: Solve

$$u'(t) = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix} u(t)$$

We need eigen pairs.

Earlier, we found: Eigenvalues are $-1, 2, 4$.

The eigenvectors assoc. with $\lambda = -1$ are the nonzero multiples of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

the eigenvectors assoc. with $\lambda = 2$ are nonzero multiples of $\begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix}$.

the eigenvectors assoc. with $\lambda = 4$ are nonzero multiples of $\begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix}$.

$\therefore$ The general sol'n is

$$u(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix} e^{4t}$$

Note: In this case

$$u(t) = \underbrace{\begin{pmatrix} -e^{-t} & -e^{2t} & 4e^{4t} \\ 0 & -\frac{3}{2}e^{2t} & \frac{5}{2}e^{4t} \\ e^{-t} & e^{2t} & e^{4t} \end{pmatrix}}_{\text{Fundamental matrix.}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Example: An initial value problem. Solve

$$\begin{cases} u'(t) = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix} u(t) \\ u(0) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{cases}$$

same ODEs as above

initial data.

The general sol'n is

$$u(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix} e^{4t}$$

Find c_1, c_2, c_3 so that

$$c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 4 \\ 0 & -3/2 & 5/2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

-1, -1, 4, -1;
0, -3/2, 5/2, 1;
1, 1, 1, 2;
The rref of the_Display_Matrix is
1, 0, 0, 32/15;
0, 1, 0, -1/3;
0, 0, 1, 1/5;

$$\Rightarrow c_1 = \frac{32}{15}, \quad c_2 = -\frac{1}{3}, \quad c_3 = \frac{1}{5}$$

The general sol'n is

$$u(t) = \frac{32}{15}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} - \frac{1}{3}\begin{pmatrix} -1 \\ -3/2 \\ 1 \end{pmatrix} e^{2t} + \frac{1}{5}\begin{pmatrix} 4 \\ 5/2 \\ 1 \end{pmatrix} e^{4t}$$

Note: If $u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ then

$$x(t) = -\frac{32}{15} e^{-t} + \frac{1}{3} e^{2t} + \frac{4}{5} e^{4t}$$

$$y(t) = \frac{1}{2} e^{2t} + \frac{1}{2} e^{4t}$$

$$z(t) = \frac{32}{15} e^{-t} - \frac{1}{3} e^{2t} + \frac{1}{5} e^{4t}$$

How do we solve $u' = Au$ when A is a real nxn matrix and A has either complex eigenvalues or not enough real eigenvectors (perhaps because some eigenvalues are repeated and we don't have enough linearly independent eigenvectors)?

Also, see the text.

(*) I'll discuss complex here. $\boxed{\text{I'll discuss the repeated ones later or in a video.}}$

For each complex eigenvalue $a + b\,i$ (and its conjugate) we have the pair of solutions

2 eig. vals $a+bi$ and $a-bi$

we expect 2 sol'n pieces to cover these.

a, b real numbers.

$$C_1\, e^{at}(\cos bt\,\vec{\mathbf{u}} - \sin bt\,\vec{\mathbf{v}})$$

$$C_2\, e^{at}(\cos bt\,\vec{\mathbf{v}} + \sin bt\,\vec{\mathbf{u}})$$

where $\vec{u} + i\vec{v}$ is an eig. vector assoc. with $a+bi$

(Here $\vec{u}$ and $\vec{v}$ are real vectors.)

These combine to give a sol'n that looks like the sum of these

$$c_1 e^{at}\left(\cos(bt)\,\vec{u} - \sin(bt)\,\vec{v}\right) + c_2 e^{at}\left(\cos(bt)\,\vec{v} + \sin(bt)\,\vec{u}\right)$$

Secret: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ has $a \pm bi$ as eigen values.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Example: Solve

$$w'(t) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} w(t)$$

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2. Find the characteristic polynomial of the coefficient matrix in the system above, and evaluate it at $\lambda = 1$.

a. 1
b. 2
c. 3
d. 4
e. None of these.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)(1-\lambda) + 4$$

$$= 1 - 2\lambda + \lambda^2 + 4$$

$$= 5 - 2\lambda + \lambda^2$$

at $\lambda = 1$ we get 4.

Eigenvalues are sol'ns to $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

$\therefore$ $\lambda = 1 + 2i$ and $\lambda = 1 - 2i$ are the eigenvalues.

$\underline{\lambda = 1+2i}$: Find the nonzero sol'ns to

$$(A-(1+2i)I)z = \vec{0}.$$

$$\left(\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} - (1+2i)I\right) z = \vec{0}$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} z = \vec{0}$$

__Note:__ $1+2i$ is an eig. value. $\therefore$ $\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}$ is singular. So, it is enough to work with the 2^{nd} row.

$$2z_1 - 2i z_2 = 0$$

$$\Rightarrow \quad z_1 = 2i z_2$$

$$\Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2i z_2 \\ z_2 \end{pmatrix} = z_2 \begin{pmatrix} 2i \\ 1 \end{pmatrix}, \quad z_2 \neq 0.$$

$\Rightarrow \begin{pmatrix} 2i \\ 1 \end{pmatrix}$ is a representative eigenvector __AND__

$$\begin{pmatrix} 2i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= \vec{u}} + i \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{= \vec{v}}.$$

$1+2i$ is the eigen value

$a = 1$

$b = 2$

let's write the sol'n. Recall the pieces are

$a=1$

$$C_1\, e^{at}(\cos bt\,\vec{\mathbf{u}} - \sin bt\,\vec{\mathbf{v}})$$

$\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$b=2$

$\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$C_2\, e^{at}(\cos bt\,\vec{\mathbf{v}} + \sin bt\,\vec{\mathbf{u}})$$

$\Rightarrow$ the general sol'n is

$$c_1 e^{t}\left(\cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) + c_2 e^{t}\left(\cos(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -2c_1 e^{t}\sin(2t) + 2c_2 e^{t}\cos(2t) \\ c_1 e^{t}\cos(2t) + c_2 e^{t}\sin(2t) \end{pmatrix} \leftarrow \text{General sol'n}$$

$$= \underbrace{\begin{pmatrix} -2e^{t}\sin(2t) & 2e^{t}\cos(2t) \\ e^{t}\cos(2t) & e^{t}\sin(2t) \end{pmatrix}}_{\text{Fundamental matrix.}}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Nonhomogeneous Problems

$$u'(t) = Au(t) + f(t)$$

Solution Process:

1. Get the general solution to $u'(t) = A\,u(t)$.

2. Get any "particular solution" to $u'(t) = A\,u(t) + f(t)$

Example: Find a particular solution to

$$x' = x - 3y - e^{t}$$
$$y' = -2x + 2y + e^{2t}$$

(Hint: Use an analog of undetermined coefficients to find the particular solution.)

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3. A
4. B
5. C