Notes:

- See the discussion board for curve information and instructions for obtaining a scanned graded copy of your midterm exam.
- Homework is posted.
- **There is no excuse for not having excellent online quiz grades!**

Open EMCF07b.

EMCF07b $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

1. One eigen value of the matrix A is 3. What is the other eigenvalue?

a. 2
b. -2
c. -1
d. 0
e. None of these.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 4$$

$$= 1 - 2\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

Eig. vals.

$\lambda = 3$, $\underline{\underline{\lambda = -1}}$.

EMCF07b $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

2. One eigenvector associated with the eigenvalue 3 of the matrix A has 1 as its first entry. What is its second entry of this eigenvector?

a. 2
b. -2
c. -1
d. 0
e. None of these.

$\lambda = 3$: Eig. vectors are nonzero sol'ns to $Ax = 3x$, which is equiv. to

$$(A - 3I)x = \vec{0}$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Aug. matrix:

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{pmatrix}$$

$R_1 + R_2 \to R_2$

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-2x_1 + 2x_2 = 0 \Leftrightarrow x_1 = x_2.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_2 \neq 0.$$

First entry $= 1 \Rightarrow x_2 = 1$

$\Rightarrow$ second entry $= 1$.

Solutions to linear first order systems.

Motivating Example: Solve

$$x' = x - 3y$$
$$y' = -2x + 2y$$

It is understood that x and y are functions of the same independent variable. e.g. $x = x(t)$
$y = y(t)$

$$x'(t) = x(t) - 3y(t)$$
$$y'(t) = -2x(t) + 2y(t)$$

we will "drop" the "of t" portion.

$$x' = x - 3y$$
$$y' = -2x + 2y$$

Matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

derivative of my unknown

unknown

Names: $u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$

$\Rightarrow$ the system is $\boxed{u' = Au.}$

Let's see how eigenvalues and eigenvectors play a role in solving this type of problem.

$$\boxed{u' = Au.}$$

Motivation: If A is just a real number then

$$u(t) = Ce^{At}$$

But, it is not. Still, you might think the exponential function plays a role. It does!!

Guess: A sol'n could look like

$$u = \vec{v}e^{kt}$$

If so, $u' = Au$

$$\Longleftrightarrow \quad \vec{v}k\cancel{e^{kt}} = A\vec{v}\cancel{e^{kt}}$$

$$A\vec{v} = k\vec{v}$$

If $\vec{v} \neq \vec{0}$ then $\vec{v}$ is an eigenvector assoc. with the eig. value k.

Back to

$$x' = x - 3y$$
$$y' = -2x + 2y$$

$u = \begin{pmatrix} x \\ y \end{pmatrix}$

$$u' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} u$$

Eigen pairs of $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$.

eigen values and associated eigen vectors

The characteristic polynomial of A is given by
det(the_Display_Matrix - z I) = -4 - (3) z + z^2
The eigenvalues of A are the roots, which are -1 and 4.

A - (-1)I is

$$\begin{pmatrix} 2, & -3; \\ -2, & 3; \end{pmatrix}$$

The augmented matrix for (A-(-1)I)x=0

$$\begin{pmatrix} 2, & -3, & 0; \\ -2, & 3, & 0; \end{pmatrix}$$

The rref is

$$\begin{pmatrix} 1, & -3/2, & 0; \\ 0, & 0, & 0; \end{pmatrix} \to \quad x_1 = \frac{3}{2}x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix},$$

$\lambda = -1$

$x_2 \neq 0$

A - (4)I is

$$\begin{pmatrix} -3, & -3; \\ -2, & -2; \end{pmatrix}$$

The augmented matrix for (A-4I)x=0 is

$$\begin{pmatrix} -3, & -3, & 0; \\ -2, & -2, & 0; \end{pmatrix}$$

The rref is

$$\begin{pmatrix} 1, & 1, & 0; \\ 0, & 0, & 0; \end{pmatrix} \to \quad x_1 = -x_2 \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$\lambda = 4$

$x_2 \neq 0$

$\therefore \quad \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$ are

sol'ns to

$$x' = x - 3y$$
$$y' = -2x + 2y$$

$\Leftrightarrow u' = Au$

Linearity

So what?

You can show that

$$u = c_1 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

is the general sol'n to the system of ODEs.

Recall: $u = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$

$$x = \frac{3}{2} c_1 e^{-t} - c_2 e^{4t}$$
$$y = c_1 e^{-t} + c_2 e^{4t}$$

Here c_1 and c_2 are arbitrary constants.

Solve the Initial Value Problem

system of ODEs

$$\begin{cases} x' = x - 3y \\ y' = -2x + 2y \end{cases}$$

$$\left.\begin{aligned} x(0) &= -1 \\ y(0) &= 2 \end{aligned}\right\} \text{Initial Data}$$

$$x = \frac{3}{2}c_1 e^{-t} - c_2 e^{4t}$$

$$y = c_1 e^{-t} + c_2 e^{4t}$$

Goal: Find c_1, c_2 so that the initial data is satisfied.

$$\left[\begin{aligned} \tfrac{3}{2}c_1 - c_2 &= -1 \\ \boxed{c_1 + c_2 = 2} \end{aligned}\right]$$

$c_1 = 2 - c_2$

$\frac{3}{2}(2 - c_2) - c_2 = -1$

$3 - \frac{5}{2}c_2 = -1$

$\frac{5}{2}c_2 = 4$

$c_2 = \frac{8}{5}$

$c_1 = \frac{2}{5}$

$\therefore$

$$x = \frac{3}{5}e^{-t} - \frac{8}{5}e^{4t}$$

$$y = \frac{2}{5}e^{-t} + \frac{8}{5}e^{4t}$$

First Order Linear Systems of Differential Equations

$$u'(t) = Au(t)$$

A is an $n \times n$ real matrix.
A is known.
$u(t)$ is an $n \times 1$ unknown function.

Goal: Find $u(t)$.

Terms: General solution, initial value problem, fundamental matrix.

to $u'(t) = Au(t)$

An expression for $u(t)$ in terms of n arbitrary constants, so that whenever $u(t_0)$ is specified, we can find unique values of the arb. constants so that $u(t)$ satisfies this data.

given

$$\begin{cases} u'(t) = Au(t) \\ u(t_0) = \vec{u}_0 \end{cases}$$

system of ODEs

initial data.

known

known

A fundamental matrix for $u'(t) = Au(t)$ is a matrix $U(t)$ which is nonsingular and has columns that solve $u'(t) = Au(t)$.

Note: If $U(t)$ is a fundamental matrix then the general sol'n to $u' = Au$ is

$$U(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Note: $\begin{array}{l} x' = x - 3y \\ y' = -2x + 2y \end{array}$ has a fundamental matrix given by $\begin{pmatrix} \frac{3}{2}e^{-t} & -e^{4t} \\ e^{-t} & e^{4t} \end{pmatrix}$

Question: Is there a relation to higher order scalar linear differential equations?

A: Yes

charact poly is $\lambda^2 - 2\lambda - 3$

e.g. Consider $y'' - 2y' - 3y = 0.$

Note: If we rename

$v = y$ and $w = y'$

Then
$$\begin{cases} v' = y' = w \\ w' = y'' = 3y + 2y' = 3v + 2w \end{cases}$$

$$\begin{cases} v' = w \\ w' = 3v + 2w \end{cases}$$

companion matrix

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Note: characteristic poly of $\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$

$$\det\begin{pmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = -\lambda(2-\lambda) - 3$$
$$= \lambda^2 - 2\lambda - 3$$

In General...

special case

How do we solve $u' = Au$ when A is a real nxn matrix with n distinct real eigenvalues?

Setting: $\lambda_1, \lambda_2, \ldots, \lambda_n$ distinct real eigen values

$\updownarrow \quad \updownarrow \quad \updownarrow$

These will be linearly independent. $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ representative eigen vectors.

Term: Fundamental Matrix

$$U(t) = \begin{pmatrix} \vec{u}_1 e^{\lambda_1 t} & \vec{u}_2 e^{\lambda_2 t} & \cdots & \vec{u}_n e^{\lambda_n t} \end{pmatrix}$$

AND the general sol'n is given by

$$U(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{c_1 \vec{u}_1 e^{\lambda_1 t} + c_2 \vec{u}_2 e^{\lambda_2 t} + \cdots + c_n \vec{u}_n e^{\lambda_n t}}$$

5 min

Example: Solve

$$u'(t) = \underbrace{\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}}_{A} u(t)$$

First: Find eigen pairs.

Charact poly: $\det(A - \lambda I)$

$$= \det \begin{pmatrix} 2-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \\ 2 & -2 & 1-\lambda \end{pmatrix}$$

A is now
2, 2, 3;
1, 2, 1;
2, -2, 1;
The characteristic polynomial of A is given by
det(A - z I) = -8 + (-2) z + (5) z^2 + (-1) z^3
The eigenvalues of A are
-1.0000000000000013 + 0 i $\longleftrightarrow \lambda=-1$
4.000000000000001 + 0 i $\longleftrightarrow \lambda=4$
1.9999999999999993 + 0 i $\longleftrightarrow \lambda=2$
(1)A - (-1)I is
3, 2, 3;
1, 3, 1;
2, -2, 2;
the_Display_Matrix is now
3, 2, 3, 0;
1, 3, 1, 0;
2, -2, 2, 0;
The rref of the_Display_Matrix is
1, 0, 1, 0;
0, 1, 0, 0;
0, 0, 0, 0;

$\lambda=-1 \qquad x_2=0 \qquad x_1=-x_3$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-x_3\\0\\x_3\end{pmatrix}=x_3\begin{pmatrix}-1\\0\\1\end{pmatrix},\ x_3\neq 0$$

(1)A - (4)I is
-2, 2, 3;
1, -2, 1;
2, -2, -3;
the_Display_Matrix is now
-2, 2, 3, 0;
1, -2, 1, 0;
2, -2, -3, 0;
The rref of the_Display_Matrix is
1, 0, -4, 0;
0, 1, -5/2, 0;
0, 0, 0, 0;

$\lambda=4 \qquad x_2=\frac{5}{2}x_3$

$x_1=4x_3$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}4x_3\\\frac{5}{2}x_3\\x_3\end{pmatrix}=x_3\begin{pmatrix}4\\5/2\\1\end{pmatrix},\ x_3\neq 0$$

(1)A - (2)I is
0, 2, 3;
1, 0, 1;
2, -2, -1;
the_Display_Matrix is now
0, 2, 3, 0;
1, 0, 1, 0;
2, -2, -1, 0;
The rref of the_Display_Matrix is
1, 0, 1, 0;
0, 1, 3/2, 0;
0, 0, 0, 0;

$\lambda=2 \qquad x_2=-\frac{3}{2}x_3$

$x_1=-x_3$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-x_3\\-\frac{3}{2}x_3\\x_3\end{pmatrix}=x_3\begin{pmatrix}-1\\-\frac{3}{2}\\1\end{pmatrix},$$

$x_3\neq 0$.

$\therefore$ a fundamental matrix is

$$U(t)=\begin{pmatrix}-e^{-t} & 4e^{4t} & -e^{2t}\\ 0 & \frac{5}{2}e^{4t} & -\frac{3}{2}e^{2t}\\ e^{-t} & e^{4t} & e^{2t}\end{pmatrix}$$ AND

the general sol'n is

$$U(t)\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=c_1\begin{pmatrix}-e^{-t}\\0\\e^{-t}\end{pmatrix}+c_2\begin{pmatrix}4e^{4t}\\\frac{5}{2}e^{4t}\\e^{4t}\end{pmatrix}+c_3\begin{pmatrix}-e^{2t}\\-\frac{3}{2}e^{2t}\\e^{2t}\end{pmatrix}$$

Example: Solve the initial value problem

$$\begin{cases} u'(t) = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix} u(t) \\ u(0) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{cases}$$

The ODE we just viewed

Initial data.

General sol'n

$$\begin{pmatrix} -e^{-t} & 4e^{4t} & -e^{2t} \\ 0 & \frac{5}{2}e^{4t} & -\frac{3}{2}e^{2t} \\ e^{-t} & e^{4t} & e^{2t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

We need

$$\begin{pmatrix} -1 & 4 & -1 \\ 0 & 5/2 & -3/2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

The augmented matrix is
-1, 4, -1, -1;
0, 5/2, -3/2, 1;
1, 1, 1, 2;
The rref is
1, 0, 0, 32/15;
0, 1, 0, 1/5;
0, 0, 1, -1/3;

$$\Rightarrow c_1 = \frac{32}{15},\ c_2 = \frac{1}{5},\ c_3 = -\frac{1}{3}$$

$\therefore$

$$u(t) = \frac{32}{15}\begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 4e^{4t} \\ \frac{5}{2}e^{4t} \\ e^{4t} \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -e^{2t} \\ -\frac{3}{2}e^{2t} \\ e^{2t} \end{pmatrix}$$

Note: If $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then

$$x = -\frac{32}{15}e^{-t} + \frac{4}{5}e^{4t} + \frac{1}{3}e^{2t}$$

$$y = \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t}$$

$$z = \frac{32}{15}e^{-t} + \frac{1}{5}e^{4t} - \frac{1}{3}e^{2t}$$

How do we solve $u' = Au$ when A is a real nxn matrix and A has either complex eigenvalues or not enough real eigenvectors (perhaps because some eigenvalues are repeated and we don't have enough linearly independent eigenvectors)?

I'll discuss complex here. I'll discuss the repeated ones later or in a video.

For each complex eigenvalue $a + b\,i$ (and its conjugate) we have the pair of solutions

a, b real numbers

2 min.

2 eig. vals $a+bi$ and $a-bi$

we expect 2 sol'n pieces to cover these.

2 linearly ind. sol'ns

$$C_1\, e^{at}\left(\cos(bt)\vec{\mathbf{u}} - \sin(bt)\vec{\mathbf{v}}\right)$$

$$C_2\, e^{at}\left(\cos(bt)\vec{\mathbf{v}} + \sin(bt)\vec{\mathbf{u}}\right)$$

Here $\vec{u}$ and $\vec{v}$ are real vectors, and $\vec{u}+i\vec{v}$ is an eig. vector assoc. with the eig. value $a+bi$.

General sol'n: Sum of thes.

$$C_1\, e^{at}\left(\cos(bt)\vec{\mathbf{u}} - \sin(bt)\vec{\mathbf{v}}\right) + C_2\, e^{at}\left(\cos(bt)\vec{\mathbf{v}} + \sin(bt)\vec{\mathbf{u}}\right)$$

where C_1, C_2 are arb. constants.

Example: Solve

$$x'(t) = 2x(t) + y(t)$$
$$y'(t) = -x(t) + 2y(t)$$

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3. Find the characteristic polynomial of the coefficient matrix in the system above, and evaluate it at $\lambda = 1$.

a. 1
b. 2
c. 3
d. 4
e. None of these.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 + 1$$

Eval at $\lambda = 1 \Rightarrow$

$1^2 + 1 = 2$

Roots of $(2-\lambda)^2+1$:

Solve $(2-\lambda)^2 + 1 = 0$

$(2-\lambda)^2 = -1$

$2-\lambda = \pm i$

$\Rightarrow \lambda = 2 \mp i$

Eig. vals: $\lambda = 2+i$, $\lambda = 2-i$

$\lambda = 2+i$: Solve for nonzero sol'ns of

$$(A - (2+i)I)z = \vec{0}$$

$$\left(\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2+i & 0 \\ 0 & 2+i \end{pmatrix} \right) z = \vec{0}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Aug. matrix: $\begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{pmatrix}$

$iR_1 + R_2 \to R_2$ $\begin{pmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$-iz_1 + z_2 = 0 \iff z_2 = iz_1$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ iz_1 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad z_1 \neq 0.$$

representative eig. vector

Note: $\begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{u}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{v}}.$

$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 2+i \qquad a=2,\ b=1$

General sol'n:

$$C_1\, e^{at}(\cos(bt)\vec{\mathbf{u}} - \sin(bt)\vec{\mathbf{v}}) + C_2\, e^{at}(\cos(bt)\vec{\mathbf{v}} + \sin(bt)\vec{\mathbf{u}})$$

$$= c_1 e^{2t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + c_2 e^{2t}\left(\cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} c_1 e^{2t}\cos(t) + c_2 e^{2t}\sin(t) \\ -c_1 e^{2t}\sin(t) + c_2 e^{2t}\cos(t) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{2t}\cos(t) & e^{2t}\sin(t) \\ -e^{2t}\sin(t) & e^{2t}\cos(t) \end{pmatrix}}_{\text{Fundamental matrix.}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

What if we do not have enough linearly independent eigenvectors?

Example:
$$x'(t) = 11x(t) - 25y(t)$$
$$y'(t) = 4x(t) - 9y(t)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 11 & -25 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If $u = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$u' = \begin{pmatrix} 11 & -25 \\ 4 & -9 \end{pmatrix}^{=A} u.$$

11, -25;
4, -9;
The characteristic polynomial of A is given by
det(A - z I) = 1 - (2) z + z^2 = (z-1)^2
The only eigenvalue of A is 1, and it is repeated.
A - (1)I is
10, -25;
4, -10;
The augmented matrix for (A - I)z=0
10, -25, 0;
4, -10, 0;
The rref is
1, -5/2, 0;
0, 0, 0;

$\lambda = 1$

$$\Rightarrow z_1 = \tfrac{5}{2} z_2 \Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} z_2 \\ z_2 \end{pmatrix} = z_2 \begin{pmatrix} \frac{5}{2} \\ 1 \end{pmatrix}, \quad z_2 \neq 0.$$

Taking $z_2 = 2$ gives $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ as a rep. eig. vector.

Hm... we get one column.

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} e^t$$

So, we need a strategy to get a second lin. ind sol'n.

Let's look for a second sol'n in the form $(\vec{v}t + \vec{w})e^t$

We need $\left((\vec{v}t + \vec{w})e^t\right)' = A(\vec{v}t + \vec{w})e^t$

$$\vec{v}\cancel{e^t} + (\vec{v}t + \vec{w})\cancel{e^t} = A\vec{v}t\cancel{e^t} + A\vec{w}\cancel{e^t}$$

$$\vec{v} + \vec{w} + t\vec{v} = tA\vec{v} + A\vec{w}$$

i.e. $A\vec{v} = \vec{v}$ and $A\vec{w} = \vec{w} + \vec{v}$ $\iff$ $A\vec{v} = \vec{v}$, $(A - I)\vec{w} = \vec{v}$

Try $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $\Leftarrow$ $\vec{v}$ is an eig. vector assoc. with $\lambda = 1$.

$$(A - I)\vec{w} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 10 & -25 \\ 4 & -10 \end{pmatrix} \vec{w} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -25 \\ 4 & -10 \end{pmatrix} \vec{w} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -25 & 5 \\ 4 & -10 & 2 \end{pmatrix}$$

10, -25, 5;
4, -10, 2;
The rref of the_Display_Matrix is
1, -5/2, 1/2;
0, 0, 0;

$$\Rightarrow w_1 = \frac{5}{2} w_2 + \frac{1}{2}$$

$$\Rightarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} w_2 + \frac{1}{2} \\ w_2 \end{pmatrix}$$

. Use any w_2 value. I just need $\vec{w}$ that works. Use $w_2 = 0$

$\vec{w} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$. Recall $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

2nd Sol'n: $(\vec{v}t + \vec{w})e^t = \begin{pmatrix} (5t + \frac{1}{2})e^t \\ 2te^t \end{pmatrix}$

Putting this with $\begin{pmatrix} 5 \\ 2 \end{pmatrix} e^t$ gives 2 Lin. Ind. Sol'ns. $\Rightarrow$

Fundamental Matrix

$$U(t) = \begin{pmatrix} 5e^t & (5t + \frac{1}{2})e^t \\ 2e^t & 2te^t \end{pmatrix}$$

General Sol'n is

$$U(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} 5e^t \\ 2e^t \end{pmatrix} + c_2 \begin{pmatrix} (5t + \frac{1}{2})e^t \\ 2te^t \end{pmatrix}.$$

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4. A
5. B
6. C