Notes:

- Email yflores@math.uh.edu to obtain a scanned graded copy of your midterm exam.
- The median score on the midterm was 81.
- Homework is posted.
- **There is no excuse for not having excellent online quiz grades!**

Open EMCF07.

EMCF07 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

1. One eigen value of the matrix A is 3. What is the other eigenvalue?

Possible solutions.

-1

1. Characteristic polynomial.

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 4$$

The eigenvalues are the $\underline{\text{roots}}$!

Set $(1-\lambda)^2 - 4 = 0$.

$$\lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda = 3$ or $\boxed{\lambda = -1}$

2. The product of the eigenvalues (up to multiplicity) is the determinant.

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 - 4 = -3$$

One eigenvalue is 3. The other is λ_0.

$$3 \cdot \lambda_0 = -3$$
$$\Rightarrow \lambda_0 = -1.$$

3. The sum of the eigenvalues (up to multiplicity) is the trace of the matrix.

$\underbrace{\text{trace of the matrix}}_{\text{sum of the diagonal entries.}}$

In our case $\text{Trace}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 + 1 = 2$.

$\therefore$ $3 + \lambda_0 = 2$ $\Rightarrow \lambda_0 = -1$.

Up to multiplicity, an nxn matrix has exactly n eigenvalues!!

EMCF07 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

2. One eigenvector associated with the eigenvalue 3 of the matrix A has 1 as its first entry. What is the ~~its~~ second entry of this eigenvector?

1 ?

$\lambda = 3$: An eigenvector $\vec{x}$ is a nonzero vector solving

$$Ax = 3x$$

$$\iff (A - 3I_2)\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix}\vec{x} = \vec{0}$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Aug. matrix: $\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{pmatrix} \xrightarrow{R_1 + R_2 \to R_2} \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$-2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where x_2 is any nonzero real number.

examples of eig. vectors assoc. with $\lambda = 3$:

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 2.315 \\ 2.315 \end{pmatrix}, \ldots$

Back to the question: 1 is the first entry. $\Rightarrow$

$$x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \cancel{x_2} \end{pmatrix} \Rightarrow x_2 = 1$$

Solutions to linear first order systems.

of differential equations.

Motivating Example: Solve

$$\boxed{\begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}}$$

here, x and y are functions of a common independent variable. Let's use t.

$x \equiv x(t)$, $y \equiv y(t)$.

A solution to this system is a vector function of the form $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ where $x(t), y(t)$ solve the system.

Let's try to solve the system:

$$x' = x - 3y$$

$$y' = -2x + 2y$$

Note: This is

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

* Idea * Spse we give some new names.

$$u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$$

Then our system is $\boxed{\frac{d}{dt} u = Au}$

If A was a number (which it is not) you might say

$$u = Ce^{At}$$

unfortunately, A is not a number. But this form DOES play a role.

we can find individual solutions that look like

$$\boxed{u(t) = e^{ct}\vec{v}}$$

where c is a number and $\vec{v}$ is a vector.

Let's see this in our case.

$$u'(t) = Au(t)$$

$$ce^{ct}\vec{v} = A\underbrace{e^{ct}}_{\text{number for each value of } t}\vec{v}$$

$$\cancel{e^{ct}} \cdot c\vec{v} = \cancel{e^{ct}} \cdot A\vec{v}$$

ie. $A\vec{v} = c\vec{v}$

If $\vec{v} \neq \vec{0}$ then $\vec{v}$ is an eigenvector assoc. with the eigenvalue c.

Let's see how eigenvalues and eigenvectors play a role in solving this type of problem.

From above $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ can have the form $\boxed{e^{ct}\vec{v}}$ where c is an eigen value of $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$ with assoc. eigenvector $\vec{v}$.

Eigenvalues of $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$.

Solve $\det(A - \lambda I_2) = 0$.

i.e. I find the roots of the characteristic polynomial.

$$\det\begin{pmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{pmatrix} = 0 \iff (1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0 \iff \lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0.$$

$$\lambda = 4, \ \lambda = -1.$$

Eigen vectors:

$\lambda = 4$: we need nonzero vectors $\vec{v}$ so that

$$A\vec{v} = 4\vec{v}$$

i.e. $$(A - 4I_2)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix}\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Aug. Matrix

$$\begin{pmatrix} -3 & -3 & 0 \\ -2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, if $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

$$\Rightarrow \vec{v} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

where $v_2 \neq 0$.

Every eigenvector assoc. with $\lambda = 4$ is a nonzero scalar multiple of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\therefore$ $c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a sol'n to

$$\boxed{\begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}}$$

$$\boxed{\begin{pmatrix} -c_1 e^{4t} \\ c_1 e^{4t} \end{pmatrix} \equiv \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}$$

Note: $C_1 = 0$ is ok too.

$\lambda = -1$: Eig. vectors are nonzero vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfying

$$A\vec{v} = -\vec{v}$$

i.e.

$$(A + I_2)\vec{v} = \vec{0}.$$

$$\boxed{A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Aug. matrix:

$$\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2v_1 - 3v_2 = 0 \Rightarrow v_1 = \frac{3}{2} v_2$

$$\Rightarrow v = \begin{pmatrix} \frac{3}{2} v_2 \\ v_2 \end{pmatrix} = \frac{1}{2} v_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

where $v_2 \neq 0$.

$\therefore$ the eigenvectors assoc. with $\lambda = -1$ are nonzero scalar multiples of $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$\rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_{12} e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{solves} \quad \begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}$$

i.e. $\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3C_{12} e^{-t} \\ 2C_{12} e^{-t} \end{pmatrix}}$ where C_{12} is an arbitrary scalar.

$\therefore$ $c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $c_2 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ both solve $\begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}$. Here c_1 and c_2 are arbitrary real #s.

Observation: Linearity.

LHS $\equiv$ derivative
RHS $\equiv$ matrix vector multiplication
} both linear

$\therefore$ the sum of any 2 sol'ns will be a sol'n.

In fact,

$$c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is the general solution to $\begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}$.

i.e. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -c_1 e^{4t} + 3c_2 e^{-t} \\ c_1 e^{4t} + 2c_2 e^{-t} \end{pmatrix}$.

Infinitely many sol'ns. One for each choice of c_1 and c_2.

Back to

$$\begin{aligned} x' &= x - 3y \\ y' &= -2x + 2y \end{aligned}$$

Observation: Solutions were found by finding the eigenvalues and eigenvectors of the coefficient matrix from the right hand sides of the equations.

Solve the Initial Value Problem

$$\begin{pmatrix} x' = x - 3y \\ y' = -2x + 2y \end{pmatrix} \leftarrow \text{system of ODEs}$$

$$\left. \begin{matrix} x(0) = -1 \\ y(0) = 2 \end{matrix} \right) \leftarrow \text{initial data.}$$

general sol'n:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -c_1 e^{4t} + 3c_2 e^{-t} \\ c_1 e^{4t} + 2c_2 e^{-t} \end{pmatrix}$$

use the initial data.

$$x(0) = -1 \iff -c_1 + 3c_2 = -1$$

$$y(0) = 2 \iff c_1 + 2c_2 = 2$$

$$\begin{pmatrix} -1 & 3 & -1 \\ 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 3 & -1 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{c_2 = \tfrac{1}{5}}$$

$$-c_1 + 3c_2 = -1$$

$$-c_1 + \tfrac{3}{5} = -1$$

$$\Rightarrow -c_1 = -\tfrac{8}{5}$$

$$\boxed{c_1 = \tfrac{8}{5}}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\frac{8}{5} e^{4t} + 3 \cdot \frac{1}{5} \cdot e^{-t} \\ \frac{8}{5} e^{4t} + 2 \cdot \frac{1}{5} \cdot e^{-t} \end{pmatrix} = \underbrace{\begin{pmatrix} -e^{4t} & 3e^{-t} \\ e^{4t} & 2e^{-t} \end{pmatrix}}_{\text{is a fundamental matrix.}} \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$$

First Order Linear Systems of Differential Equations

$$u'(t) = Au(t)$$

A: $n \times n$ matrix. known

$u(t)$: unknown n-component vector valued function.

Goal: Find $u(t)$.

A solution is a vector function $u(t)$ that satisfies the system.

Terms: General solution, initial value problem, $\boxed{\text{fundamental matrix.}}$

General solution: the most general solution. i.e., a solution formulation that includes all possible sol'ns.

Initial value problem:

$$u'(t) = Au(t)$$
$$u(t_0) = \vec{u}_0$$

t_0, $\vec{u}_0$: known. $u(t_0) = \vec{u}_0$: initial conditions

$U(t)$ is a fundamental matrix for the system $u'(t) = Au(t)$ iff

1. each column of $U(t)$ solves $u'(t) = Au(t)$
2. the columns are linearly independent.

Fact: One fundamental matrix is given by e^{At} where e^{At} is the matrix exponential.

Google "exponential of a matrix".

Question: Is there a relation to higher order scalar linear differential equations?

all can be rewritten as first order systems.

Answer: Yes.

$y(t)$

Illustrative Example: $\boxed{y'' - 2y' - 3y = e^{2t}}$

Note: There is an assoc. characteristic polynomial

$\boxed{r^2 - 2r - 3}$

Let's do something: (crazy)

$w = y$ $\quad \boxed{w' = z}$

$z = y'$ $\quad z' = y''$

$= 3y + 2y' + e^{2t}$

$\boxed{z' = 3w + 2z + e^{2t}}$

$$\begin{pmatrix} w' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}}_{A} \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}.$$

charact. poly of A: $\det(A - \lambda I_2)$

$$= \det \begin{pmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = \boxed{\lambda^2 - 2\lambda - 3}$$

Special Case...

How do we solve $u' = Au$ when A is a real nxn matrix with n distinct real eigenvalues?

Spse A has distinct eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$

FYI: In this case, the eigenvectors will all be linearly independent.

with associated eigenvectors

$\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n$.

$$u(t) = c_1 e^{\lambda_1 t}\vec{V}_1 + c_2 e^{\lambda_2 t}\vec{V}_2 + \cdots + c_n e^{\lambda_n t}\vec{V}_n$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real numbers.

Term: Fundamental Matrix

$$= \underbrace{\begin{pmatrix} e^{\lambda_1 t}\vec{V}_1 & e^{\lambda_2 t}\vec{V}_2 & \cdots & e^{\lambda_n t}\vec{V}_n \end{pmatrix}}_{\substack{n\times n \text{ matrix.} \\ \text{Fundamental matrix.}}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

vector vector vector

Example: Solve

$$u'(t)=\begin{pmatrix}2 & 2 & 3\\ 1 & 2 & 1\\ 2 & -2 & 1\end{pmatrix}u(t)$$

A is now
2, 2, 3;
1, 2, 1;
2, -2, 1;
The characteristic polynomial of A is given by det(A - z I) = -8 + (-2) z + (5) z^2 + (-1) z^3
The roots are the eigen values.
The eigenvalues of A are
-1.0000000000000013 + 0 i
4.000000000000001 + 0 i
1.9999999999999993 + 0 i
i.e. eigenvalues are -1, 4, and 2.

lambda = -1
(1)A - (-1)I is
3, 2, 3; 0
1, 3, 1; 0
2, -2, 2; 0
rref
1, 0, 1; 0
0, 1, 0; 0
0, 0, 0; 0

$$(A-(-1)I_3)\vec{v}=\vec{0}$$

$$\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$

$$\Rightarrow \quad v_2=0$$

$$v_1+v_3=0 \qquad v_1=-v_3$$

$$\vec{v}=\begin{pmatrix}-v_3\\0\\v_3\end{pmatrix}=v_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$\lambda=-1.$

lambda=4
(1)A - (4)I is
-2, 2, 3; 0
1, -2, 1; 0
2, -2, -3; 0
rref
1, 0, -4; 0
0, 1, -5/2; 0
0, 0, 0; 0

$$(A-4I_3)\vec{v}=\vec{0}$$

$$\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$

$$v_2=\tfrac{5}{2}v_3 \qquad v_1=4v_3$$

$$\vec{v}=\begin{pmatrix}4v_3\\\tfrac{5}{2}v_3\\v_3\end{pmatrix}=\tfrac{1}{2}v_3\begin{pmatrix}8\\5\\2\end{pmatrix}$$

$\lambda=4$

lambda=2
(1)A - (2)I is
0, 2, 3; 0
1, 0, 1; 0
2, -2, -1; 0
rref
1, 0, 1; 0
0, 1, 3/2; 0
0, 0, 0; 0

$$(A-2I_3)\vec{v}=\vec{0}$$

$$v_1+v_3=0 \qquad v_1=-v_3$$

$$v_2+\tfrac{3}{2}v_3=0 \qquad v_2=-\tfrac{3}{2}v_3$$

$$\vec{v}=\cdots=\tfrac{1}{2}v_3\begin{pmatrix}-2\\-3\\2\end{pmatrix}$$

$\lambda=2$

$$\therefore \quad u(t)=c_1e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix}+c_2e^{4t}\begin{pmatrix}8\\5\\2\end{pmatrix}+c_3e^{2t}\begin{pmatrix}-2\\-3\\2\end{pmatrix}$$

where c_1, c_2, c_3 are arb. scalars.

Example: Solve the initial value problem

$$\begin{cases} u'(t) = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix} u(t) \\ u(0) = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{cases}$$

← we just solved.

← initial data.

$$u(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$

at $t=0$

$$c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

i.e.

$$\begin{pmatrix} -1 & 8 & -2 \\ 0 & 5 & -3 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

-1, 8, -2, -1;
0, 5, -3, 1;
1, 2, 2, 2;
The rref of the_Display_Matrix is
1, 0, 0, 32/15;
0, 1, 0, 1/10;
0, 0, 1, -1/6;

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 32/15 \\ 1/10 \\ -1/6 \end{pmatrix}.$$

$$u(t) = \frac{32}{15} e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{10} e^{4t} \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + \frac{-1}{6} e^{2t} \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{32}{15} e^{-t} + \frac{4}{5} e^{4t} + \frac{1}{3} e^{2t} \\ \frac{1}{2} e^{4t} + \frac{1}{2} e^{2t} \\ \frac{32}{15} e^{-t} + \frac{1}{5} e^{4t} - \frac{1}{3} e^{2t} \end{pmatrix}.$$

How do we solve $u' = Au$ when A is a real nxn matrix and A has either complex eigenvalues or not enough real eigenvectors (perhaps because some eigenvalues are repeated and we don't have enough linearly independent eigenvectors)?

I'll discuss complex here. See the text for the repeated case.

For each complex eigenvalue $a + b\,\mathrm{i}$ (and its conjugate) we have the pair of solutions

Two linearly independent solutions:

$$C_1 e^{at}\left(\cos(bt)\vec{u} - \sin(bt)\vec{v}\right)$$

$$C_2 e^{at}\left(\cos(bt)\vec{v} + \sin(bt)\vec{u}\right)$$

where $\vec{u} + i\vec{v}$ is an eigenvector associated with the eigenvalue $a + bi$, and both $\vec{u}$ and $\vec{v}$ are real vectors.

Example: Solve

$$x'(t) = 2x(t) + y(t)$$
$$y'(t) = -x(t) + 2y(t)$$

EMCF07

3. Find the characteristic polynomial of the coefficient matrix in the system above, and evaluate it at $\lambda = 1$.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

charact. poly:

$$\det(A - \lambda I_2)$$
$$= \det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix}$$
$$= (2-\lambda)^2 + 1$$

evaluate at $\lambda = 1$.

$$(2-1)^2 + 1 = 1 + 1 = 2$$

What if we do not have enough linearly independent eigenvectors?

Example:

$$x'(t) = 11x(t) - 25y(t)$$
$$y'(t) = 4x(t) - 9y(t)$$

See the OTHER summer 2012 video posted under 7/26 on the course homepage.