Information

- The final exam dates/times are posted. ← See the scheduler.
- A practice problem set is posted.
- The final exam is comprehensive, but the latter material is weighted more heavily.
- Laplace transforms will be on the final exam, and I will provide a formula sheet.
- Assignment 6 is the last homework assignment.
- Take care of your online quizzes.
- This is the last online class.

EMCF08

1. A

One more example with repeated eigenvalues:

Solve $u' = A\,u$ with

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}, \boxed{u(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}$$

Note: If $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then

2. c

$$\begin{aligned} x' &= -3x + y - z, \\ y' &= -7x + 5y - z, \\ z' &= -6x + 6y - 2z, \end{aligned} \qquad \boxed{\begin{aligned} x(0) &= 1 \\ y(0) &= 1 \\ z(0) &= -1 \end{aligned}}$$

1 Eigen pairs:

A is now
-3, 1, -1;
-7, 5, -1;
-6, 6, -2;
The eigenvalues of A are
-2.000000009856192 + 0 i
-1.999999901438086 + 0 i
4.000000000000001 + 0 i

$\leftarrow$ -2 is a repeated eigen value

$\leftarrow$ 4 is an eigenvalue.

A - (-2)I is

-1, 1, -1;
-7, 7, -1;
-6, 6, 0;

The aumented matrix for (A - (-2)I)v = 0 is

-1, 1, -1; 0
-7, 7, -1; 0
-6, 6, 0; 0

The rref is

1, -1, 0; 0 $\rightarrow v_1 = v_2$
0, 0, 1; 0 $\rightarrow v_3 = 0$
0, 0, 0; 0

representative eig. vector.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_2 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 \neq 0.$$

Note: we only get one L.I. eig. vector assoc. with $\lambda = -2$.

the augmented matrix for (A - 4I)v = 0 is given by

-7, 1, -1, 0;
-7, 1, -1, 0;
-6, 6, -6, 0;

The rref is

1, 0, 0, 0; $\rightarrow v_1 = 0$
0, 1, -1, 0; $\rightarrow v_2 = v_3$
0, 0, 0, 0;

representative eig. vector assoc. with $\lambda = 4$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 \neq 0$$

Note: There is only one L.I. eig. vector assoc. with $\lambda = 4$.

Note: Since A is 3×3, our sol'n will be

$$u(t) = c_1\begin{pmatrix}0\\1\\1\end{pmatrix}e^{4t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{-2t} + \begin{pmatrix}?\\?\end{pmatrix}e^{-2t}$$

we need a 3rd piece.

We are "short" b/c we did not have enough LI eig. vectors assoc. with $\lambda = -2$

To get another piece, try

$$(\vec{v}t + \vec{w})e^{-2t}$$

$\Leftarrow$ Find $\vec{v}$ and $\vec{w}$ so that this solves $u' = Au$ AND this sol'n is L.I. with the other pieces.

$$(\vec{v}t+\vec{w})(-2)\cancel{e^{-2t}} + \vec{v}\cancel{e^{-2t}} = A(\vec{v}t+\vec{w})\cancel{e^{-2t}}$$

$$-2\vec{v}t + (-2\vec{w} + \vec{v}) = A\vec{v}t + A\vec{w}$$

$A\vec{v} = -2\vec{v}$ $\longleftrightarrow$ $\vec{v}$ is an eigenvector assoc. with the eigenvalue -2.

$\Updownarrow$ Try $\begin{pmatrix}1\\1\\0\end{pmatrix}$ since all eig. vects. assoc. with $\lambda = -2$ are nonzero scalar mult of $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

$\updownarrow$ $A\vec{v} = \lambda\vec{v}$

$A\vec{w} = -2\vec{w} + \vec{v}$

$\vec{v} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$$A\vec{w} = -2\vec{w} + \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Solve for $\vec{w}$.

$$\underbrace{(A-(-2)I)}\vec{w} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

the augmented matrix is
-1, 1, -1, 1;
-7, 7, -1, 1;
-6, 6, 0, 0;
The rref is
1, -1, 0, 0; $\rightarrow$ $w_1 = w_2$
0, 0, 1, -1; $\rightarrow$ $w_3 = -1$
0, 0, 0, 0;

$$\Rightarrow \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} = \begin{pmatrix}w_2\\w_2\\-1\end{pmatrix}$$

Choose any w_2 value we like.

Use $w_2 = 0$

$$\Rightarrow \quad w = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$\therefore$ $\left(\begin{pmatrix}1\\1\\0\end{pmatrix}t + \begin{pmatrix}0\\0\\-1\end{pmatrix}\right)e^{-2t}$ works.

$$\Rightarrow \quad u = \underbrace{c_1\begin{pmatrix}0\\1\\1\end{pmatrix}e^{4t}} + \underbrace{c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{-2t}} + \underbrace{c_3\left(\begin{pmatrix}1\\1\\0\end{pmatrix}t + \begin{pmatrix}0\\0\\-1\end{pmatrix}\right)e^{-2t}}$$

Are these L.I.??

Each is a sol'n, so we can just check if they are L.I. at a specific t value.

Use $t=0$.

$$\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}}_{\text{L.I.?}}$$

$$\det\begin{pmatrix}0 & 1 & 0\\1 & 1 & 0\\1 & 0 & -1\end{pmatrix} = 0 + 1\cdot(-1)^{1+2}\det\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix} + 0$$

$$= (-1)(-1) = 1 \neq 0$$

$\Rightarrow$ They are L.I.

$\therefore$ the general sol'n is

$$u = c_1\begin{pmatrix}0\\1\\1\end{pmatrix}e^{4t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{-2t} + c_3\left(\begin{pmatrix}1\\1\\0\end{pmatrix}t + \begin{pmatrix}0\\0\\-1\end{pmatrix}\right)e^{-2t}$$

Recall: we want $u(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

the augmented matrix is

0, 1, 0, 1;
1, 1, 0, 1;
1, 0, -1, -1;

The rref is

1, 0, 0, 0;
0, 1, 0, 1;
0, 0, 1, 1;

$\Rightarrow$ $c_1 = 0$, $c_2 = 1$, $c_3 = 1$

$\therefore$

$$u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right) e^{-2t}$$

u(t) ≡ unknown n×1 vector function of t.

A ≡ known n×n matrix

f(t) ≡ known n×1 function

Nonhomogeneous Problems

homog. problem

$$u'(t) = Au(t) + f(t)$$

Solution Process:

1. Get the general solution $u_h(t)$ to $\underbrace{u'(t) = A\,u(t)}$. ⟷ $u' - Au = 0$

2. Get any $\underbrace{\text{"particular solution"}}$ $u_p(t)$ to $u'(t) = A\,u(t) + f(t)$

any thing that works.

3. The general solution is $u(t) = u_h(t) + u_p(t)$.

$\boxed{\text{use strategic guessing}}$

Ex.
$$x' = -2x + y + \cos(t)$$
$$y' = x - y + e^{-t} - 2$$

Then $u = \begin{pmatrix} x \\ y \end{pmatrix}$ gives

$$u' = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}}_{\equiv A} u + \underbrace{\begin{pmatrix} \cos(t) \\ e^{-t} - 2 \end{pmatrix}}_{\equiv f(t)}$$

5 min break

3. B

4. A

Example: Find the general solution to

$$x' = x - 3y - e^t$$
$$y' = -2x + 2y + e^{2t}$$

(Use an analog of undetermined coefficients to find the particular solution.)

$u = \begin{pmatrix} x \\ y \end{pmatrix}$ $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$

$u' = Au + \underbrace{\begin{pmatrix} -e^t \\ e^{2t} \end{pmatrix}}_{\text{"} f(t)}$

① Find $u_h(t)$. Note

u_h solves $u_h' = Au_h$

Find eigen pairs:

A is now
1, -3;
-2, 2;
The eigenvalues of A are -1 and 4.

the augmented matrix for (A - (-1)I)v = 0 is
2, -3, 0;
-2, 3, 0;
The rref is
1, -3/2, 0;
0, 0, 0;

$\Longrightarrow$ $v_1 = \frac{3}{2} v_2$ $\quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} v_2 \\ v_2 \end{pmatrix} = \frac{1}{2} v_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \; v_2 \neq 0$

rep. eig. vector

the augmented matrix for (A - 4I)v = 0 is
-3, -3, 0;
-2, -2, 0;
The rref is
1, 1, 0;
0, 0, 0;

$v_1 = -v_2 \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; v_2 \neq 0$

rep. eig. vector

$$\therefore \quad u_h(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$$

② Get $u_p(t)$. Look closely at the equation.

$$u' = Au + \begin{pmatrix} -e^{t} \\ e^{2t} \end{pmatrix}$$

$$u' = Au + \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{t}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}}$$

2 pieces.

Guesses:

$\underbrace{\vec{v} e^{t}} \qquad \underbrace{\vec{w} e^{2t}}$

should be ok b/c e^{t}, e^{2t} are NOT in $u_h(t)$.

Try
$u_p(t) = \vec{v} e^{t} + \vec{w} e^{2t}$
Find $\vec{v}, \vec{w}$.

Sbst u_p into $u' = Au + \begin{pmatrix}-1\\0\end{pmatrix}e^t + \begin{pmatrix}0\\1\end{pmatrix}e^{2t}$

$$\vec{v}e^t + 2\vec{w}e^{2t} = A(\vec{v}e^t + \vec{w}e^{2t}) + \begin{pmatrix}-1\\0\end{pmatrix}e^t + \begin{pmatrix}0\\1\end{pmatrix}e^{2t}$$

$$\vec{v}e^t + 2\vec{w}e^{2t} = A\vec{v}e^t + A\vec{w}e^{2t} + \begin{pmatrix}-1\\0\end{pmatrix}e^t + \begin{pmatrix}0\\1\end{pmatrix}e^{2t}$$

Need

$$\vec{v}\cancel{e^t} = A\vec{v}\cancel{e^t} + \begin{pmatrix}-1\\0\end{pmatrix}\cancel{e^t}$$

$$2\vec{w}\cancel{e^{2t}} = A\vec{w}\cancel{e^{2t}} + \begin{pmatrix}0\\1\end{pmatrix}\cancel{e^{2t}}$$

- $\vec{v} = A\vec{v} + \begin{pmatrix}-1\\0\end{pmatrix}$
- $2\vec{w} = A\vec{w} + \begin{pmatrix}0\\1\end{pmatrix}$

Solve for $\vec{v}$ and $\vec{w}$.

$$A = \begin{pmatrix}1 & -3\\-2 & 2\end{pmatrix}$$

- $(I - A)\vec{v} = \begin{pmatrix}-1\\0\end{pmatrix}$
- $(2I - A)\vec{w} = \begin{pmatrix}0\\1\end{pmatrix}$

5. D

- $\begin{pmatrix}0 & 3\\2 & -1\end{pmatrix}\vec{v} = \begin{pmatrix}-1\\0\end{pmatrix}$
- $\begin{pmatrix}1 & 3\\2 & 0\end{pmatrix}\vec{w} = \begin{pmatrix}0\\1\end{pmatrix}$

$v_2 = -\frac{1}{3}$ $2v_1 - v_2 = 0 \Rightarrow v_1 = -\frac{1}{6}$

$$\Rightarrow \vec{v} = \begin{pmatrix}-1/6\\-1/3\end{pmatrix}$$

$\hookrightarrow$ $w_1 = \frac{1}{2}$, $w_1 + 3w_2 = 0$ $\Rightarrow$ $w_2 = -\frac{1}{6}$

$$\Rightarrow \vec{w} = \begin{pmatrix} 1/2 \\ -1/6 \end{pmatrix}$$

$$u_p(t) = \vec{v}e^t + \vec{w}e^{2t} = \begin{pmatrix} -1/6 \\ 1/3 \end{pmatrix} e^t + \begin{pmatrix} 1/2 \\ -1/6 \end{pmatrix} e^{2t}$$

$\Rightarrow$ the general sol'n to

$$u' = Au + \begin{pmatrix} -e^t \\ e^{2t} \end{pmatrix}$$

is

$$u(t) = u_h(t) + u_p(t)$$

$$= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -1/6 \\ 1/3 \end{pmatrix} e^t + \begin{pmatrix} 1/2 \\ -1/6 \end{pmatrix} e^{2t}$$

where c_1 and c_2 are arbitrary constants.

6. A
7. A
8. A
9. A
10. A

Example: Give the solution to

$$\underbrace{u' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} u + \begin{pmatrix} 4e^{2t} \\ 3e^{4t} \end{pmatrix}}, \underbrace{u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

① Get the general sol'n to this.

② Satisfy the initial conditions.

Find $u_h(t)$ and $u_p(t)$.

Solves $u' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} u$

Get eigen pairs. You can show that the eigenvalues are $\lambda = 2$ and $\lambda = 4$.

$\underline{\lambda = 2:}$

the augmented matrix for (A-2I)v=0 is
1, -1, 0;
-1, 1, 0;
The rref is
1, -1, 0;
0, 0, 0;

$\Rightarrow \quad v_1 = v_2 \quad \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 \neq 0$

rep. eig. vector.

$\underline{\lambda = 4:}$

the augmented matrix for (A-4I)v=0 is
-1, -1, 0;
-1, -1, 0;
The rref is
1, 1, 0;
0, 0, 0;

$\Rightarrow \quad v_2 = -v_1 \quad \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_1 \neq 0$

rep. eig. vector.

Just for something different, we decided to solve for v2 in terms of v1. It really does not matter. The representative eigenvectors, in either case, will be scalar multiples of each other.

$$\therefore \quad u_h(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

<u>Now let's</u> get <u>$u_p(t)$</u>: Look carefully.

u_p solves
$$u' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} u + \begin{pmatrix} 4e^{2t} \\ 3e^{4t} \end{pmatrix}.$$

<u>i.e.</u>
$$u_p' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} u + \underbrace{\begin{pmatrix} 4 \\ 0 \end{pmatrix} e^{2t}} + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix} e^{4t}}$$

Guess to match

<u>Note</u>: e^{2t} and e^{4t} are pieces of $u_h(t)$. This complicates things.

$$u_p' = \begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}u + \underbrace{\begin{pmatrix}4\\0\end{pmatrix}e^{2t}} + \underbrace{\begin{pmatrix}0\\3\end{pmatrix}e^{4t}}$$

Guess to match

try $\quad$ try

$$(\vec{v}t+\vec{w})e^{2t} \qquad\qquad (\vec{p}t+\vec{q})e^{4t}$$

Here we go. Set

$$u_p(t) = (\vec{v}t+\vec{w})e^{2t} + (\vec{p}t+\vec{q})e^{4t}$$

and subst. into the ODE.

$$2(\vec{v}t+\vec{w})e^{2t} + \vec{v}e^{2t} + 4(\vec{p}t+\vec{q})e^{4t} + \vec{p}e^{4t}$$

$$= A\left[(\vec{v}t+\vec{w})e^{2t} + (\vec{p}t+\vec{q})e^{4t}\right] + \begin{pmatrix}4\\0\end{pmatrix}e^{2t} + \begin{pmatrix}0\\3\end{pmatrix}e^{4t}$$

$$e^{2t}\left[2\vec{v}t + 2\vec{w} + \vec{v}\right] + e^{4t}\left[4\vec{p}t + 4\vec{q} + \vec{p}\right]$$

$$= e^{2t}\left[A(\vec{v}t+\vec{w}) + \begin{pmatrix}4\\0\end{pmatrix}\right] + e^{4t}\left[A(\vec{p}t+\vec{q}) + \begin{pmatrix}0\\3\end{pmatrix}\right]$$

Match e^{2t} and e^{4t} terms.

$$2\vec{v}t + 2\vec{w} + \vec{v} = A(\vec{v}t+\vec{w}) + \begin{pmatrix}4\\0\end{pmatrix}$$

$$4\vec{p}t + 4\vec{q} + \vec{p} = A(\vec{p}t+\vec{q}) + \begin{pmatrix}0\\3\end{pmatrix}.$$

11. B

12. B

13. B

14. B

15. B

Look at t and non-t parts

$\underline{\underline{t}}$: $2\vec{v} = A\vec{v}$ — $\vec{v}$ is an eig. vector assoc. with $\lambda = 2$.

i.e. $\vec{v} = \alpha\begin{pmatrix}1\\1\end{pmatrix}$.

$2\vec{w} + \vec{v} = A\vec{w} + \begin{pmatrix}4\\0\end{pmatrix}$

$$(A - 2I)\vec{w} = \vec{v} - \begin{pmatrix}4\\0\end{pmatrix}$$

$A = \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}$

$$\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\vec{w} = \alpha\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}4\\0\end{pmatrix}$$

<u>Aug. matrix</u>

$$\begin{pmatrix}1 & -1 & \alpha-4\\-1 & 1 & \alpha\end{pmatrix} \xrightarrow{R_1 + R_2 \to R_2} \begin{pmatrix}1 & -1 & \alpha-4\\0 & 0 & 2\alpha-4\end{pmatrix}$$

We <u>need</u> for there ~~to~~ be a sol'n.

$\therefore$ we need $\boxed{\alpha = 2}$ for there to be a sol'n.

$$\Rightarrow \vec{v} = \alpha\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}. \quad (*)$$

$$w_1 = w_2 - 2$$

$$\Rightarrow \vec{w} = \begin{pmatrix}w_2 - 2\\w_2\end{pmatrix}$$

We just need one that works.

Take $w_2 = 2 \Rightarrow \vec{w} = \begin{pmatrix}0\\2\end{pmatrix}$. $(*)$

$$\therefore \quad u_p(t) = \left(\begin{pmatrix}2\\2\end{pmatrix}t + \begin{pmatrix}0\\2\end{pmatrix}\right)e^{2t} + \left(\vec{p}\,t + \vec{q}\right)e^{4t}$$

Let's get $\vec{p}$ and $\vec{q}$.

16 – 20 all C.

Recall:

$$4\vec{p}t + 4\vec{q} + \vec{p} = A(\vec{p}t + \vec{q}) + \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Separate the t and non-t.

t: $4\vec{p} = A\vec{p}$ — $\vec{p}$ is an eigenvector assoc. with $\lambda = 4$

i.e. $\vec{p} = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

non-t: $4\vec{q} + \vec{p} = A\vec{q} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$4\vec{q} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} = A\vec{q} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$(A - 4I)\vec{q} = \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \qquad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\vec{q} = \begin{pmatrix} \beta \\ -\beta - 3 \end{pmatrix}$$

Aug. matrix

$$\begin{pmatrix} -1 & -1 & \beta \\ -1 & -1 & -\beta - 3 \end{pmatrix} \xrightarrow{-R_1 + R_2 \to R_2} \begin{pmatrix} -1 & -1 & \beta \\ 0 & 0 & -2\beta - 3 \end{pmatrix}$$

$\therefore$ we need $-2\beta - 3 = 0 \Rightarrow$

$$\beta = -\frac{3}{2}.$$

$$\Rightarrow \vec{p} = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}$$

Let's get $\vec{q}$. From the first row

$$-q_1 - q_2 = \beta$$

i.e. $$q_1 = -\beta - q_2 = \frac{3}{2} - q_2$$

$$\Rightarrow \quad \vec{q} = \begin{pmatrix} \frac{3}{2} - q_2 \\ q_2 \end{pmatrix}$$

choose any q_2.

Take $q_2 = 0$. $\quad \vec{q} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$.

$$u_p(t) = \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) e^{2t} + \left(\begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 3/2 \\ 0 \end{pmatrix} \right) e^{4t}$$

$\therefore$ the general sol'n is

$$u = u_h + u_p$$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) e^{2t} + \left(\begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 3/2 \\ 0 \end{pmatrix} \right) e^{4t}$$

Now, match $u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\therefore \quad c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}3/2\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-1/2\\-1\end{pmatrix}$$

Solve for c_1 and c_2 to finish.

I'll let you ☺.

Then rewrite u using the values you find for c_1 and c_2.