Differential Calculus for Functions of Several Variables

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Other Information

- **WinPlot** – Download at http://math.exeter.edu/rparris/winplot.html
- **Sage** – Download and Online Version at http://www.sagemath.org
- **Geogebra** – Download at http://www.Geogebra.org
- **ScribLINK** – Collaborate Online at http://www.scriblink.com/
- Google – http://www.google.com Forms and Groups
- **Past presentations** – http://www.math.uh.edu/~jmorgan/Rice

Advertisement…

- AP Calculus Resources –
  http://www.HoustonACT.org (local organization)
  http://online.math.uh.edu/apcalculus/index.htm
  (online help and online automated quizzes)
  http://online.math.uh.edu/HoustonACT/ (Greg Kelley Power Points)
- AP Statistics Listserve – http://www.HoustonATS.org
- High School Math Contest – http://mathcontest.uh.edu
- *teach*HOUSTON – http://teachHOUSTON.uh.edu
- Online Master's Degree in Mathematics – MAM Program – http://www.math.uh.edu

Vectors – Algebraic and Geometric

Vectors in 2 and 3 Dimensions

$$R^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, a,b \in R \right\} \qquad R^2 = \left\{ ai + bj \,\middle|\, a,b \in R \right\}$$

$$R^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, a,b,c \in R \right\} \qquad R^3 = \left\{ ai + bj + ck \,\middle|\, a,b,c \in R \right\}$$

Vectors in *n* Dimensions ➡ $R^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \, x_i \in R \text{ for } i = 1,\ldots,n \right\}$

Vector Addition, Subtraction and Scalar Multiplication

Comparing Points and Vectors

Dot Product and Norm

Matrices

An $m \times n$ matrix has the form $A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$

where each entry $a_{i,j}$ is a number.

Shorthand: $A = \left(a_{i,j}\right)$

The set of real $m \times n$ matrices is denoted by $R_{m,n}$.

Derivatives: Calculus I Review

Determinant Question

Does the determinant of a square matrix have any geometric interpretation?

Interesting Investigation : Explore the result of acting a 2×2 matrix on the unit circle in the xy plane.

http://www.math.uh.edu/~jmorgan/Math6397/day01/first_try.html

Question

How can we extend this notion to the multivariable setting?

Function Notation

If U and V are sets, then the expression

$$f : U \to V$$

denotes that f is a function with domain U and range contained in V.

Definition

Let $f : R^n \to R^m$ and suppose $a \in R^n$. We say f is differentiable at a if and only if there is a matrix $A \in R_{m,n}$ and a function $E : R^n \to R^m$ such that

$$f(x) = f(a) + A(x-a) + E(x-a) \text{ for all } x \in R^n$$

$$\text{and } \frac{\|E(h)\|}{\|h\|} \to 0 \text{ as } h \to 0.$$

In this case, we denote $f'(a) = A$.

Examples

$f : R^2 \to R^3$ given by

$$f(x,y) = \begin{pmatrix} x - \cos(y) \\ xy + 1 \\ x - y\sin(x) \end{pmatrix}$$

$f : R^2 \to R$ given by

$$f(x,y) = x\sin(y) + 2y$$

$f : R \to R^3$ given by

$$f(t) = \begin{pmatrix} t - \sin(2t) \\ \cos(t) + 1 \\ t - 2\sin(t) \end{pmatrix}$$

How Do We Compute A Derivative?

Examples

$f: R^2 \to R^3$ given by $f(x,y) = \begin{pmatrix} x - \cos(y) \\ xy + 1 \\ x - y\sin(x) \end{pmatrix}$

$f: R^2 \to R$ given by $f(x,y) = x\sin(y) + 2y$

$f: R \to R^3$ given by $f(t) = \begin{pmatrix} t - \sin(2t) \\ \cos(t) + 1 \\ t - 2\sin(t) \end{pmatrix}$

Geometric Interpretation I

Parametric Functions $f: R^k \to R^n$

Illustrative Examples

$$f(t) = \begin{pmatrix} \cos(t) \\ \sin(2t) \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$$

Chain Rule

Geometric Interpretation II

Special case $f: R^n \to R$

Directional derivatives and gradients.

Illustrative Examples

$$f(x,y) = \frac{x^2}{9} + \frac{y^2}{4}$$

$$g(x,y,z) = x^2 + y^2 - z$$

The Second Derivative

Special case $f : R^n \to R$

Critical Points

Example

$f : R^2 \to R$ given by $f(x,y) = -x\sin(y) + \cos(x+y)$.
Give the first and second derivatives of f.

Gradient Fields, Differential Equations and Critical Points

General Case

Examples

$$f(x,y) = -xy\,e^{-\left(x^2+y^2\right)/2}$$

$$f(x,y) = x^4 + 2xy^2 - 3x^2y + y^4 + 2x - y - 3(x^2 + y^2)$$

(WinPlot)

Local Extrema and Saddle Points

Illustrative Example: $f : R^2 \to R$ given by $f(x, y) = -xy\,e^{-\left(x^2+y^2\right)/2}$

Example

$f : R^2 \to R$ given by $f(x, y) = -xy\,e^{-\left(x^2+y^2\right)/2}$.
Find the critical points of f.

Example

Use a surface plot and level curves to visualize the critical values of

$$f(x, y) = -xy\,e^{-\left(x^2+y^2\right)/2}$$

Using *Determinants* to Classify Critical Points

Example

$f: R^2 \to R$ given by $f(x,y) = -xy\, e^{-\left(x^2+y^2\right)/2}$.
Classify the critical points of f.

Newton's Method

The setting $f: R \to R$

The setting $f: R^n \to R^n$

Examples

Find and classify the critical points of
$f(x,y,z) = 2x^2 + y^2 + z^2 + xy - 2xz + 3x + y - 6$

Find and classify the critical points of
$f(x,y) = x^4 + 2xy^2 - 3x^2 y + y^4 + 2x - y - 3(x^2 + y^2)$

(WinPlot and Sage)

Example

Use Newton's method to approximate a solution to

$$\begin{pmatrix} 3x - \cos(yz) = 0.5 \\ x^2 - 81(y+0.1)^2 + \sin(z) = -1.06 \\ e^{-xy} + 20z = \dfrac{10\pi - 3}{3} \end{pmatrix} \text{ near } (0,0,0).$$