Irrational phase synchronization

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We study the occurrence of physically observable phase locked states between chaotic oscillators and rotors in which the frequencies of the coupled systems are irrationally related. For two chaotic oscillators, the phenomenon occurs as a result of a coupling term which breaks the 2π invariance in the phase equations. In the case of rotors, a coupling term in the angular velocities results in very long times during which the coupled systems exhibit alternatively irrational phase synchronization and random phase diffusion. The range of parameters for which the phenomenon occurs contains an open set, and is thus physically observable.

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In recent years, synchronization of coupled chaotic systems has been a topic of much interest. Different types of chaotic synchronization have been studied theoretically [1], and observed in nature [2], controlled laboratory experiments [3], and space extended or infinite dimensional systems [4].

Phase synchronization (PS) of chaos refers to the phenomenon in which the phases of two interacting systems evolve in step with each other, even when the corresponding amplitudes are only weakly correlated [5]. PS is frequently considered to be the chaotic counterpart of phase locking between periodic systems.

$m{:}n$ phase locking of periodic oscillators (PL) has been studied since the 17th century [6]. It can be thought of as the appearance of a stable limit cycle on the invariant torus defined by the cross product of the phases φ_1, φ_2 of the two coupled oscillators. If the two systems are phase locked, any 2π advance in φ_1 is accompanied by a corresponding $(m/n)2\pi$ advance in φ_2 (m,n being integer numbers). An equivalent statement can be formulated in terms of ψ_1 and ψ_2, the lifts of the two phases to the real line. If these lifts satisfy $|\psi_2-(m/n)\psi_1|<C$, with C a positive constant, the two systems are said to be phase locked.

Similarly $m{:}n$ frequency locking (FL) occurs when the systems adjust their mean frequencies ω_1 and ω_2 so that they satisfy $\omega_2=(m/n)\omega_1$. For a wide class of coupled nonlinear oscillators this condition is satisfied in a finite region of parameter space called the resonance tongue [9]. It is well known that for nonlinear periodic oscillators resonance tongues corresponding to irrational frequency ratios ($\omega_1 = r\omega_2$, with r irrational) are of zero measure.

In periodic systems PL is not a threshold phenomenon, in the sense that even two uncoupled periodic oscillators can exhibit PL if their frequencies happen to be rational multiples of each other. The situation is different in the case of chaotic or noisy periodic systems. Due to the phase diffusion, two uncoupled systems may exhibit only FL, but not PL. In this case PL is a genuine threshold phenomenon. We use this distinction to argue that irrational phase synchronization is a nontrivial state in coupled chaotic systems. In particular, we demonstrate that for chaotic systems it is meaningful to consider states where $\Delta_r\psi=|\psi_1-r\psi_2|$ (with r an irrational number) oscillates around a constant value C with amplitude smaller than π.

To motivate this discussion, consider a pair of round plates coupled by a belt running along their perimeter. Assume that the belt does not slip around the first plate, but may slip as it moves around the second, and that the first plate is rotating uniformly. If the radii of the two plates are irrationally related we can assume that the force on the second plate is proportional to the difference in its angular velocity and the velocity of the belt, so that $\ddot{\psi}_2=-\epsilon(\dot{\psi}_2-r\dot{\psi}_1)$, where r is an irrational number. We ignore inertia in this simplified approach. Integrating this equation we obtain $\dot{\psi}_2 = -\epsilon(\psi_2-r\psi_1)+\epsilon H+\dot{\psi}_2(0)$, where $H=\psi_2(0)-r\psi_1(0)$. It is easy to see that the lifts of the phases approach the relation $\psi_2=r\psi_1+H-\Delta_r\Psi/\epsilon$, where $\Delta_r\Psi=\dot{\psi}_2(0)-r\dot{\psi}_1(0)$. Thus a 2π advance in ψ_1 corresponds to a $r2\pi$ advance in ψ_2. A similar computation shows that if $\ddot{\psi}_2=-\epsilon(\dot{\psi}_2-r\dot{\psi}_1)+\eta(t)$, with $\eta(t)<\Gamma$ is some bounded, integrable function, the same conclusion follows, up to an error bounded by Γ/ε. Thus irrational phase synchronization is possible in this system even if the system is noisy. Note that the appearance of a term $\epsilon(\psi_2-r\psi_1)$ in a two-dimensional (2D) dynamical system means that there exists a force that depends on the relative velocities, commonly associated with dissipative viscous forces found in mechanical systems. In addition, to the previous mechanical example, such a viscous term can also be found in two pendula attached end to end [7], or in two pistons attached to moving cylinders. Such type of dissipative viscous forcing term can also be used to control oscillations in a shipboard crane, as it was done in Ref. [8].

There are a few important features that distinguish this case from examples of coupled oscillators typically found in the literature. In the present case, there is no preferred phase difference between the two rotating plates, the asymptotic state being determined by the initial conditions. Moreover, the phase locked state does not appear in a saddle node bifurcation, and hence the phase slips typical of noisy phase locked oscillators [1,5] will not have the same origin in this case.

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Following this example, we say that two systems exhibit r phase synchronization (r-PS) if phase variables can be defined such that their lifts to the line ψ_1 and ψ_2 satisfy $|\psi_1 - r\psi_2 - C| = |\Delta_r \psi - C| < K < \pi$. In the present example the condition $K < \pi$ is essential, since otherwise the phase of one system would not tell us anything about the relation of the phases in the original systems, and the two systems could not be called synchronous [10].

A more concrete example is provided by a pair of coupled chaotic Rössler oscillators [11]

$$\dot{x}_{1,2} = -\omega_{1,2} y_{1,2} - z_{1,2},$$

$$\dot{y}_{1,2} = \omega_{1,2} x_{1,2} + a y_{1,2},$$

$$\dot{z}_{1,2} = f + z_{1,2}(x_{1,2} - c), \tag{1}$$

where $a = 0.15, f = 0.2, c = 10$. The systems are in the chaotic, phase coherent regime, with approximate angular frequencies ω_1 and ω_2. Rewriting the equations in cylindrical coordinates ($x_{1,2} \equiv \varrho_{1,2}\cos\psi_{1,2}, y_{1,2} \equiv \varrho_{1,2}\sin\psi_{1,2}$), and adding a term coupling the phases of the two systems we obtain

$$\dot{\varrho}_{1,2} = -z_{1,2}\cos(\psi_{1,2}) + a\varrho_{1,2}\sin^2(\psi_{1,2}),$$

$$\dot{\psi}_{1,2} = \omega_{1,2} + a\cos(\psi_{1,2})\sin(\psi_{1,2}) + \frac{z_{1,2}}{\varrho_{1,2}}\sin(\psi_{1,2}) + \varepsilon_{1,2}F_{1,2}(\psi_1, \psi_2),$$

$$\dot{z}_{1,2} = f + z_{1,2}[\varrho_{1,2}\cos(\psi_{1,2}) - c], \tag{2}$$

where ϱ_1 (ϱ_2) and ψ_1 (ψ_2) are the amplitude and phase of the first (second) oscillator, $\varepsilon_{1,2}$ are two real parameters controlling the strength of the coupling, and $F_{1,2}(\psi_1, \psi_2): R^2 \to R$ are coupling functions specified below. Couplings affecting only the phases of oscillators are used in models of physical systems such as Josephson junctions [12] and phase locked loops [13].

A difficulty in showing irrational phase synchronization in system (2) is due to the fact that in numerical simulations all quantities are represented by rational numbers. We consider $\omega_1 = 1$, $\omega_2 = 1.3$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and $F_1(\psi_1, \psi_2) = -F_2(\psi_1, \psi_2) = \varepsilon(2r_n\psi_2 - \psi_1)$, where $r_n \equiv a_n / a_{n+1}$, and a_n is the nth element of the Fibonacci series. As a result $\{r_n\}$ is a series of rational numbers converging to the golden mean ($\lim_{n\to\infty} r_n = \sigma \sim 0.6180$).

Figure 1 summarizes the main results of the simulations of system (2). At $\varepsilon = 0.004$ the phase differences $\Delta_{2r_n} \equiv \psi_1(t) - 2r_n\psi_2(t)$ exhibit small oscillations around a constant value for each n [Fig. 1(a)]. Furthermore, for $n \geqslant 5$ the phase differences Δ_{2r_n} oscillate around a value that is essentially independent of n. This has been checked up to $n = 20$, corresponding to the maximum precision allowed by our computer. Figure 1(b) shows the third and fourth Lyapunov exponents in the spectrum versus coupling strength ε at $n = 20$. $2r$-PS occurs when the Lyapunov exponent which is initially zero becomes negative. This occurs before the smallest positive Lyapunov exponent becomes negative [for comparison see Fig. 1(d)] and is in agreement with observations in the case of $n:m$ phase synchronization [14]. In Figs. 1(c) and 1(d) we graph the third and fourth (the first and second) Lyapunov exponents in the spectrum versus the index n at $\varepsilon = 0.04$ [marked with an arrow in Fig. 1(b)], showing that their values become independent of n, as n increases. This is consistent with $2r$-PS characterized by the condition $|\psi_1(t) - 2r\psi_2(t) - C| < \pi$ (with $r = \sigma$), and the presence of two positive, one zero, and one negative Lyapunov exponents in the spectrum.

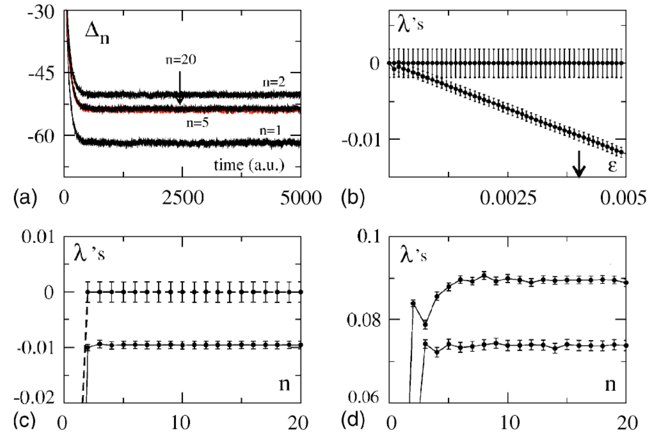

FIG. 1. Numerical integration of system (2). (a) Time evolution of $\Delta_{2r_n} \equiv \psi_1(t) - 2r_n\psi_2(t)$ for different values of n at $\varepsilon = 0.004$ [marked with an arrow in b)]. (b) Third and fourth Lyapunov exponents in the spectrum vs the coupling strength ε at $n = 20$. (c) Third and fourth Lyapunov exponents in the spectrum vs the index n at $\varepsilon = 0.04$. (d) First and second Lyapunov exponents in the spectrum vs the index n at $\varepsilon = 0.04$. Units are dimensionless.

Note that at $\varepsilon = 0$ the right-hand side of system (2) is invariant under 2π phase translations. The coupling term breaks this invariance. Hence the present system differs from the ones studied in Ref. [14], since it cannot be transformed directly back into rectangular coordinates. Given system (2) with $F_1(\psi_1, \psi_2) = -F_2(\psi_1, \psi_2) = \varepsilon(2r\psi_2 - \psi_1)$, let $\psi_1 - 2r\psi_2 = \Delta_{2r}\psi$ so that

$$\Delta_{2r}\psi' = (\varepsilon + 2r\varepsilon)(\Delta_{2r}\psi) + \omega_1 - 2r\omega_2 + g(\cos(\psi_{1,2}), \sin(\psi_{1,2}), z_{1,2}, \varrho_{1,2}), \tag{3}$$

where g is easily determined from the original equations. The numerics show that z and ϱ remain bounded, which may also be proved using a Lyapunov function argument. Therefore, there exists a constant L such that $|g| < L$ for all time. It follows that

$$|e^{(\varepsilon+2r\varepsilon)t}\Delta_{2r}\psi(t) - \Delta_{2r}\psi(0)| \leqslant \int_0^t e^{(\varepsilon+2r\varepsilon)s}|\Delta_{2r}\omega + L| ds \leqslant \frac{|\Delta_{2r}\omega + L|(e^{(\varepsilon+2r\varepsilon)t} - 1)}{-(\varepsilon + 2r\varepsilon)},$$

where $\Delta_{2r}\omega = \omega_1 - 2r\omega_2$. Therefore

$$|\Delta_{2r}\psi| \leqslant \frac{|L + \Delta_{2r}\omega|(1 - e^{-(\varepsilon+2r\varepsilon)t})}{\varepsilon + 2r\varepsilon} + e^{-(\varepsilon+2r\varepsilon)t}\Delta_{2r}\psi(0).$$

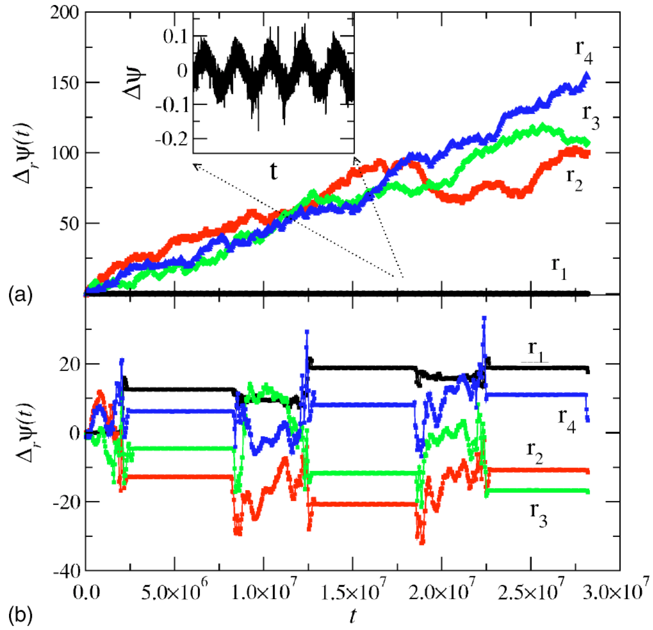

FIG. 2. Temporal behavior of $\Delta_{r_n}\psi$ for $\omega_2=0.5$ (a) and $\omega_2=0.499\,999\,390\,009\,6$ (b) for $r_1=1$, $r_2=1.047$, $r_3=1.047\,197\,55$, and $r_4=1.047\,197\,551\,196\,59$. In (a) PS is lost as r is approaching $\pi/3$. In (b), PS is robust within an interval of r values whose boundary is very close to $\pi/3$. The coupling is $\varepsilon=20$. Units are dimensionless.

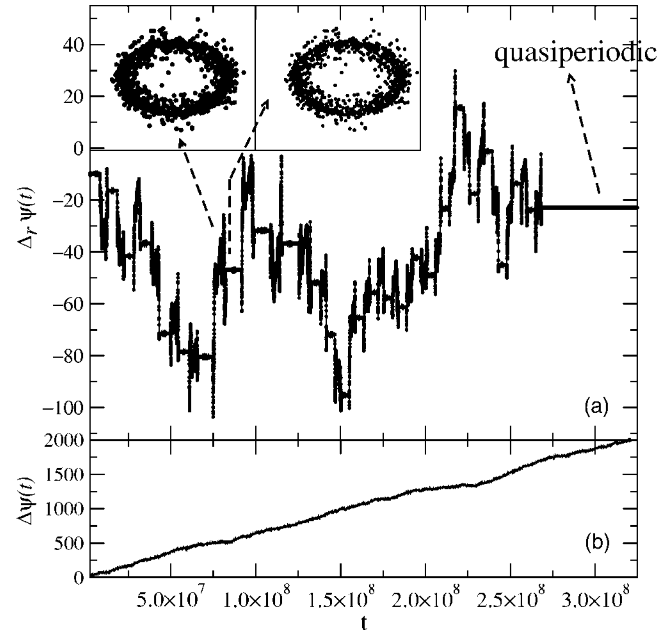

FIG. 3. Temporal evolution of $\Delta_r\psi$ for $\Delta\omega=2.099\,904\times10^{-7}$ (a) and for $\Delta\omega=10^{-3}$ (b). The two boxes in (a) show stroboscopic mappings of the attractor $\dot{\psi}_1\times\ddot{\psi}_1$ for time intervals where diffusive (left box) and regular (right box) behavior in $\Delta\psi$ is observed. $\varepsilon=20$. Notice the very large time scale $(\sim3\times10^8)$ over which the transient dynamics takes place. Units are dimensionless.

We can conclude that $\Delta_{2r}\psi$ will always be bounded, no matter what ε is. However, a small ε implies the possibility of large excursions and a slow decay of $\Delta_{2r}\psi$. Therefore $2r$-PS will occur only for sufficiently strong coupling. Note also that if we defined $\Delta_{2\tilde{r}}\psi=\psi_1-2\tilde{r}\psi_2$ for $\tilde{r}\neq r$, then Eq. (3) will contain an additional term which will grow approximately linearly. It is easy to see that such a term will lead to slow linear growth in $\Delta_{2\tilde{r}}\psi$ so that the two systems exhibit $2r$-PS for a unique value of r, in a set containing an open subset of parameter space.

Since the coupling in system (2) breaks the 2π periodicity of the right-hand side of the equations, it is not clear how to interpret such a coupling physically. We next consider a more realistic system composed of a pair of chaotic rotors, described by

$$\ddot{\psi}_{1,2}+\gamma_{1,2}\dot{\psi}_{1,2}+f_{1,2}(\psi_{1,2})=F_{1,2}(t)\pm\varepsilon(r\dot{\psi}_{2,1}-\dot{\psi}_{1,2}),\quad(4)$$

where $f_{1,2}(\psi)=\exp\{10[\cos(\psi)-1]\}\sin(8\psi)$, $F_{1,2}(t)=\alpha_{1,2}+\beta_{1,2}\sin(\omega_{1,2}t)$. Equation (4) can model resistively coupled Josephson junctions, subject to external currents of dc (ac) components $\alpha_{1,2}$ $(\beta_{1,2})$ [15]. The state variable ψ represents the angle (phase variable) while $\dot{\psi}$ represents the rotation velocity (angular frequency). In the following we set $\gamma_1=\gamma_2=0.1$, $\beta_1=1.03$, $\beta_2=1$, $\alpha_1=\alpha_2=0.01$, and $\omega_1=0.5$.

In Ref. [16] it was argued that system (4) exhibits 1:1 PS in a special set of parameters, so that $|\psi_2-\psi_1|$ remains bounded for all time. We are interested in r-PS, with r irrational, i.e., a state in which $|\Delta_r\psi-C|<\pi$, with $\Delta_r\psi=\psi_1-r\psi_2$. Following the ideas given above, we show that a r-PS state persists within an interval of r values given by $[a,b]$ with a rational and b irrational, and $|a-b|\ll1$, i.e., we demonstrate irrational PS by showing the robustness of r-PS within a parameter interval whose open boundary is an irrational (up to the numerical resolution of our computer).

We first set $\omega_2=0.5$, $\varepsilon=20$, and vary r within the interval $[1,\pi/3]$, with $\pi/3$ approximated to 15 digits. Figure 2 shows that, as we change r from a rational number $(r_1=1)$ toward an irrational one by selecting a sequence of values $r_2<r_3<\cdots<r_n<r_{n+1}<\pi/3$, the rational PS disappears. The inset represents a magnification of the evolution of $\Delta_r\psi$ for case $r=r_1$. For $\omega_2=0.499\,999\,390\,009\,6$ the behavior of $\Delta_r\psi$ is very different, as illustrated in Fig. 2(b). In this case for all r within the interval $[1,\pi/3]$, the phase difference $\Delta_r\psi(t)$ behaves intermittently, alternating between epochs of approximately constant behavior, the *laminar phase* or *plateau*, and epochs of diffusive behavior. During the intervals of diffusive behavior, the phase difference evolves apparently randomly to another closeby plateau. Setting $r\approx\pi/3$, Fig. 3(a) shows that this intermittent phenomenon persists over a very long time interval, with plateaus of approximately the same length. Finally a quasiperiodic state is reached in which the phases exhibit nonchaotic r-PS. It must be highlighted that only this latter state rigorously satisfies our condition for r-PS. Nevertheless, also during the chaotic transient the laminar phases persist over intervals much larger than the characteristic time scale of the system (the phase difference temporarily satisfy the condition for phase locking in intervals of time of the order of a million rotations). As a consequence, such a transient behavior can be observed experimentally as a transient phase synchronization regime [17].

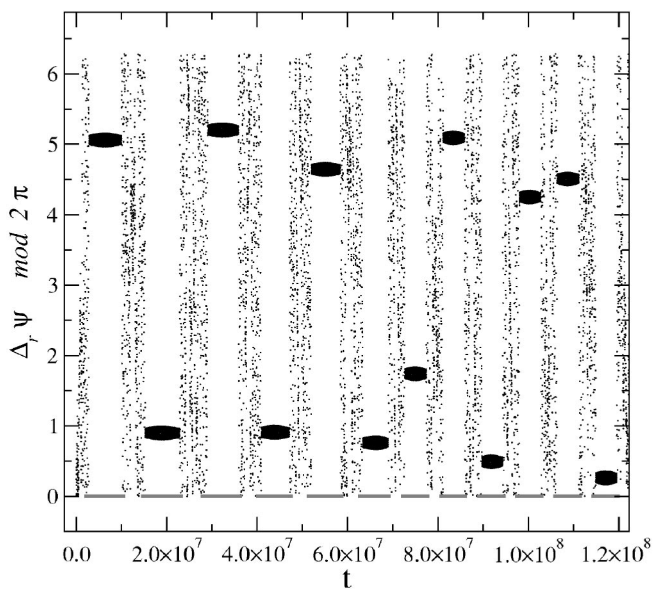

FIG. 4. The phase difference $\Delta_r\Psi$ mod 2π, for $\Delta\omega = 0.000\,000\,5$, and the broken gray thick line below represents when the maximum of the function $|\sin(\omega_1 t)-\sin(\omega_2 t)|$ is higher than 1. Units are dimensionless.

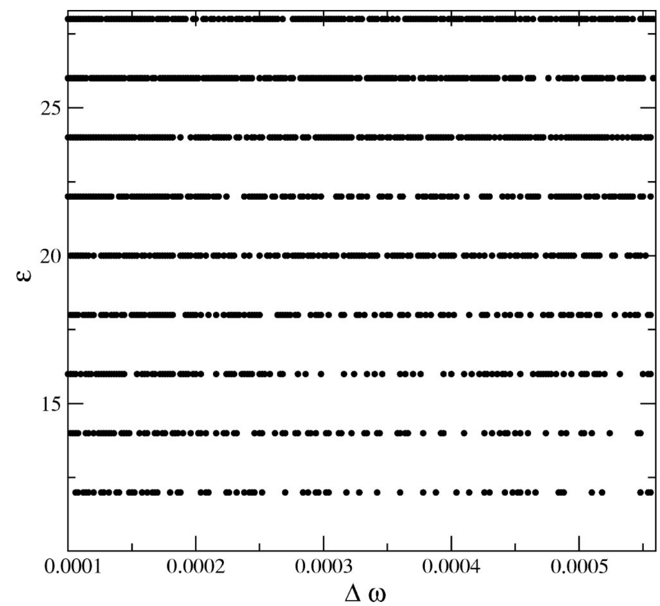

FIG. 5. Parameter space $\Delta\omega$ vs ε. Points represents parameters for which a plateau of length $\tau = 100\,000$ is found. In this picture we set $\gamma_1=\gamma_2=0.1$, $\beta_1=1.03$, $\beta_2=1$, $\alpha_1=\alpha_2=0.01$, and $\omega_1=0.5$. Note that the horizontal axis represents $\Delta\omega$ which is equal to $|\omega_1-\omega_2|$. Units are dimensionless.

Next we analyze the dependence of the laminar phases on the forcing frequency mismatch $\Delta\omega=|\omega_2-\omega_1|$ when $r \approx \pi/3$. Our aim is to show that there exists an interval of values $\Delta\omega$ for which the system exhibits the type of switching between laminar and diffusive behavior illustrated in Fig. 3(a). For $\varepsilon=20$, this scenario is observed for all choices of $\Delta\omega$ within the interval $[10^{-7},10^{-5}]$. Indeed, increasing the frequency mismatch to $\Delta\omega=10^{-3}$ results in unbounded evolution of the phase difference [see Fig. 3(b)]. Increasing ε results in increasing the size of the $\Delta\omega$ interval over which r-PS can be found. We have verified that the very same scenario persists for a large set of parameters given by $\Delta\omega=[10^{-8},10^{-6}]$, $\epsilon=[10,30]$, $\beta_2=0.9$, and $\beta_1=[0.87,0.89]$.

In Fig. 3(a), the top small insets show stroboscopic reconstructions of the $\dot{\psi}_1 \times \ddot{\psi}_1$ attractor, for diffusive phase difference (left) and for a plateau (right). Both attractors are typical of chaotic motion, and are neither quasiperiodic nor periodic.

Successive plateaus are characterized by $|\Delta_r\psi - C| < \pi$ with different, uncorrelated values of C. This means that the transient behavior is not characterized by a preferred phase difference in the system, equivalent to the system of coupled rotating plates given in the introduction.

Next, we analyze the dependence of the plateau length on $\Delta\omega$. In Fig. 4 we show that the phase difference $\Delta_r\psi$ mod 2 π, at $\Delta\omega=5\times10^{-7}$, exhibits plateaus precisely when $|\sin(\omega_1 t)-\sin(\omega_2 t)|$ is above a certain threshold (in this case the threshold is 1). Performing an analysis similar to the one leading to Eq. (4), we can conclude that $\Delta_r\psi$ can be expected to exhibit diffusive behavior, exactly when $|\sin(\omega_1 t) - \sin(\omega_2 t)|$ exceeds a critical value, in agreement with numerical observations. From this analysis it follows that the length of the laminar phase is inversely proportional to the frequency difference $|\omega_1-\omega_2|$. In fact, we have found numerically that the average laminar period, $\langle T_{lam}\rangle$, scales as $\langle T_{lam}\rangle \cong B\Delta\omega^{-1}$, with $B=0.203\pm0.006$. We emphasize that phase synchronization during the laminar chaotic states occurs only as a transient phenomenon, on time scales much larger than the time scale of the oscillations [17,18], and the r-PS condition is therefore only satisfied intermittently. The r-PS condition is satisfied fully only during the final nonchaotic state, indicating that full r-PS emerges when the system settles onto its quasiperiodic attractor.

The r-PS phenomenon observed in Eqs. (4) has some interesting features. The transient chaotic r-PS state leads to a final quasiperiodic state. Although the transient state can be very long, it invariably terminates in a r-PS quasiperiodic state. Phenomenologically, this suggests that the transient, chaotic r-PS state corresponds to a chaotic saddle in the phase space of the system. This leads us to analyze the parameter space to detect such transient r-PS states. Since the transients can be very long, we assume that whenever a plateau is observed a final quasiperiodic state is reached, occasionally checking that this is indeed the case [19].

The result are shown in Fig. 5, a parameter space for $\Delta\omega=[10^{-4},5.6\times10^{-3}]$ and $\varepsilon=[10,28]$, with the resolution of 10 points in the vertical axis and 200 points in the horizontal axis. For small values of $\Delta\omega$, the r-PS regions are dense, and therefore, were not shown in this figure. We note that there are many open areas where r-PS is observed. As ε is increased the region in which r-PS exists becomes more dense, showing that the coupling is responsible for the r-PS phenomena.

In conclusion, we have presented examples of coupled chaotic systems which exhibit irrational phase synchronization. This type of synchrony is characterized by the emergence of a phase locked state with irrational frequency ratio. In particular, we have shown that, unlike in the case of periodic systems, irrational phase synchronization is physically meaningful concept for coupled chaotic systems. Similar results can be expected for noisy periodic systems. For chaotic oscillators, such a phenomenon is induced by a coupling term involving the difference in the lifts of the phases which breaks the 2π invariance in the phase equations. A more realistic example is given by coupled chaotic rotors, where a coupling term in the angular velocities can result in alternating epochs of temporary irrational phase synchronization (chaotic r-PS) with phase diffusion, eventually leading to the setting of a quasiperiodic state where irrational phase synchronization holds. The range of parameters for which this phenomenon occurs contains an open set, and is thus physically observable.

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[19] We use the following method to detect plateaus longer than 100 000 units within a time interval of length $t=2\,000\,000$. We consider two measures: an average of the phase difference $\langle\Delta\Psi(t)\rangle=1/\tau\int_{u=t-\tau}^{u=t}\Delta\Psi(u)du$, and the phase difference. We consider two instants in time, $t=t_0$ and $t=t_0+\tau$, with $\tau=100\,000$. A plateau and hence transient, chaotic r-PS is detected, when both conditions are satisfied: (i) $|\langle\Delta\Psi(t_0+\tau)\rangle-\langle\Delta\Psi(t_0)\rangle|<K$, where $K=0.5<\pi$, and also (ii) $|\Delta\Psi(t_0+\tau)-\Delta\Psi(t_0)|<K$. Note that if the definition of r-PS is satisfied, then condition (i) should also be satisfied.