Invariant Manifolds and Synchronization of Coupled Dynamical Systems

Krešimir Josić*
Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16801
(Received 22 September 1997)

Synchronization between coupled chaotic systems can be described in terms of invariant manifolds. If such manifolds possess the additional property of normal k-hyperbolicity, it can be deduced that synchronization will persist under perturbations. This suggests a mathematical framework within which the different aspects of synchronization can be discussed and analyzed. Using these techniques, it can be shown that unidirectionally and bidirectionally coupled synchronized systems are locally equivalent. [S0031-9007(98)05621-X]

PACS numbers: 05.45.+b, 02.40.Sf

In the last decade, the surprising phenomenon of synchronization between coupled chaotic systems has generated much interest. The seminal papers of Afraimovich *et al.* [1], Fujisaka and Yamada [2], and Pecora and Carroll [3] have led to the discovery of a number of systems exhibiting various types of synchronization: system-subsystem synchronization (sometimes referred to as master-slave synchronization) [3,4], synchronization in unidirectionally and bidirectionally coupled systems [5,6], antiphase synchronization [6,7], and generalized synchronization [8,9]. This phenomenon has been observed in mechanical and electric systems [1,10], laser systems [11], biological systems [12], and Josephson junctions [13], and has also been applied in control theory [14] and for the purpose of safe communications [15]. Despite the active interest, there still remains a need for a satisfactory mathematical framework within which the various aspects of this phenomenon can be discussed and examined. The approach described in this paper provides a description of synchronization in terms of invariant manifolds. In addition to providing a unified view of the various sorts of synchronization mentioned above, it also allows one to easily prove two new results, namely, the stability of synchronization under small perturbations and the local equivalence of unidirectionally and bidirectionally coupled synchronized systems. Although invariant manifolds have been used in the study of chaotic synchronization [5,8,16], the use of k-hyperbolicity is new. To avoid technical difficulties, the ideas presented in this paper are not the most general possible. Brief mention of the possible extensions will be made.

Definitions and examples.—Roughly speaking, chaotic synchronization means that, given two coupled chaotic dynamical systems (which may or may not be identical), there exists a smooth and invertible map ϕ which carries trajectories on the attractor of the first system to trajectories on the attractor of the second, with the property that if an orbit of the first system approaches a trajectory $\mathbf{x}_1(t)$ on the attractor of the first system, then the corresponding orbit of the second system approaches the trajectory $\mathbf{x}_2(t) = \phi(\mathbf{x}_1(t))$. Thus, once transients have died away, a knowledge of the state of the first system allows one to predict the state of the second.

As stated in the introduction, a unifying approach to chaotic synchronization can be based on the theory of invariant manifolds. In order to capture the ideas in the previous paragraph, we require that these manifolds possess several additional properties. The first of these is required to make sense of the idea of the attractor for each of the two subsystems after they are coupled. *A priori*, the phase space of the coupled system will be the product of the phase spaces of the subsystems, and, if the coupled system has an attractor at all, it may be of much higher dimension than the attractors of the original subsystems. A sufficient condition to ensure that we can still speak of the attractors of the two n-dimensional subsystems is to require that the invariant manifold be the graph of a diffeomorphism $\phi: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Such an n manifold with boundary will be referred to as "*diagonal-like*." If Π^1 and Π^2 are the projections from the phase space of the coupled system onto the phase spaces of the first and second subsystems, respectively, and Ω is the attractor of the coupled system which lies inside an invariant *diagonal-like* manifold M, then the requirement that M is diagonal-like implies that there exists a smooth map between the sets $\Pi^1(\Omega)$ and $\Pi^2(\Omega)$ in the phase space of the system $\mathbf{x}_1(t)$ and $\mathbf{y}_2(t)$, respectively. This restriction can be relaxed to require that the projections Π^i be only finite to one as happens in the case of generalized synchronization [8,9].

Second, we need to require that solutions in the neighborhood of the invariant manifold also synchronize. In other words, the manifold under consideration should not be invariant only under the flow, but trajectories in some neighborhood should also be attracted to it. Such manifolds will be called "*locally attracting*." One way to ensure that an invariant manifold is locally attracting is to require it to be normally hyperbolic [17,18] with trivial unstable normal bundle.

Finally, in order to avoid some pathologies like trajectories leaving the region under consideration, we require the invariant manifolds to be compact and inflowing. An "*inflowing*" manifold is a compact manifold with boundary, such that on the boundary the vector field is directed into the interior of the manifold.

Now consider a pair of coupled dynamical systems

$$\mathbf{x}_1' = f(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_2' = g(\mathbf{x}_1, \mathbf{x}_2), \tag{1}$$

where $f, g: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are smooth functions. Synchronization between the systems $\mathbf{x}_1$ and $\mathbf{x}_2$ is defined in the following way:

Definition 1.—The systems $\mathbf{x}_1$ *and* $\mathbf{x}_2$ synchronize *if there exists a compact, diagonal-like, smooth n manifold M with boundary which is invariant under the flow, inflowing, and locally attracting. M will be referred to as the synchronization manifold.*

In particular, the synchronization manifold M can be viewed as the graph of the function ϕ defined above, and so the existence of the manifold M implies the existence of the function ϕ. As will be shown, this definition encompasses a variety of different physical situations in which chaotic synchronization has been shown to occur. In addition, in order for such synchronization to be of physical interest, it should persist under small perturbations of either the coupling or the two subsystems. The above definition leads to a natural description of sufficient conditions that ensure such stability.

The dynamics on the synchronization manifold can (and, in general, will) be quite complicated. In particular, it may have both positive and negative Lyapunov exponents. In order to ensure that synchronization persists under perturbation of the system, we require that the rate at which trajectories are attracted toward the manifold is greater than the rates of contraction or expansion within the manifold. If the rate of attraction is k times greater than the expansion or contraction rates within the manifold, the synchronization manifold is called *normally k-hyperbolic*. For a precise formulation of normal k-hyperbolicity, see [17,18]. For the present discussion, the crucial fact is that normally k-hyperbolic invariant manifolds persist as smooth manifolds under small perturbations of the underlying dynamical system (see [18] for a proof) and this leads to the following:

Definition 2.—The synchronization of x and y is called stable *if the synchronization manifold M is normally k-hyperbolic for some* $k \geq 1$.

Example 1.—Consider the case of two coupled Lorenz systems:

$$x_i' = \sigma_i(y_i - x_i) + c_i(x_2 - x_1),$$
$$y_i' = \rho_i x_i - y_i - x_i z_i + c_i(y_2 - y_1),$$
$$z_i' = -\beta_i z_i + x_i y_i + c_i(z_2 - z_1),$$

where $i = 1, 2$ and $c_1 = -c_2$. When the coefficients σ_i, ρ_i, and β_i of the two systems agree, then it is immediate that the diagonal $D \in \mathbb{R}^6$ (i.e., the set defined by $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$) is an invariant set. Using an extension of the argument given in [19], it is possible to show that there is an invariant inflowing ellipsoid E around the origin in $\mathbb{R}^6$.

Therefore, all solutions of the coupled systems remain bounded and we may take $M = D \cap E$ as a compact, inflowing, invariant 3-manifold. To show that M is, in fact, attracting, consider the difference variables $\xi_1 = x_1 - x_2$, $\xi_2 = y_1 - y_2$, and $\xi_3 = z_1 - z_2$, so that

$$\xi_1' = \sigma(\xi_2 - \xi_1) - 2c_1\xi_1,$$
$$\xi_2' = \rho\xi_1 - (1 + 2c_1)\xi_2 - x_2\xi_3 - z_1\xi_1,$$
$$\xi_3' = -(\beta + 2c_1)\xi_3 + y_1\xi_1 + x_2\xi_2.$$

The synchronization manifold M corresponds to $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$ and these coordinates are taken in a direction orthogonal to M. Therefore, showing that the synchronization manifold is stable is equivalent to showing that the origin in ξ space is attracting for all values of (x_1, y_1, z_1). For a sufficiently strong coupling c, it can be shown that

$$V = -\frac{g}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

is a Lyapunov functional for the ξ variables. The stronger the coupling, the larger the constant g that can be chosen in the expression for V. Since the Lyapunov functional is quadratic, this implies that trajectories near M are attracted exponentially, and an appropriate choice of coupling constant will make M normally k-hyperbolic. For strong coupling, the synchronization between the two systems is stable. In particular, it is not necessary that σ_i, ρ_i, and β_i be equal—slight mismatches between the parameters will not destroy synchronization.

Example 2.—System-subsystem synchronization (also known as master-slave synchronization) has generated much attention since it was first introduced in [3]. In this case, a system of differential equations $x' = f(x)$, with $x = (x_1, \ldots, x_n)$, is split into two parts; $u = (x_1, \ldots, x_{(n-m)})$ and $v = (x_{(n-m+1)}, \ldots, x_n)$, leading to the following pair of coupled systems:

$$u' = f(u, v),$$
$$v' = g_1(u, v), \qquad \hat{v}' = g_2(u, \hat{v}).$$

To extend the definitions of the previous section to this case, a dummy variable $\hat{u} = u$ can be introduced. The new variable evolves as u and is introduced to increase the dimension of the phase space to $2n$. The definitions of synchronization now apply to the coupled systems (u, v) and $(\hat{u}, \hat{v})$ which evolve in $L = \{y \in \mathbb{R}^{2n} \mid y_i = y_{i+n}, \text{ for } i = 1, \ldots, m\} \subset \mathbb{R}^{2n}$.

The original question whether the system-subsystem exhibits synchronization for $g_1 = g_2$ now becomes a question of the stability of the diagonal $D \subset L$. The generalized Lyapunov numbers of a manifold are the analog of Lyapunov exponents of trajectories in this setting. The generalized Lyapunov number α [17] measures the rate of contraction or expansion under the flow in the direction transversal to the manifold, and can, therefore, be used to replace the somewhat vague notion of *conditional Lyapunov exponents* [3]. A sufficient condition for synchronization to occur is that this generalized Lyapunov number of D be negative.

Consider the Lorenz system with a system-subsystem coupling given by

$$x' = \sigma(y - x),$$
$$y' = \rho x - y - xz, \qquad \hat{y}' = \rho x - \hat{y} - x\hat{z},$$
$$z' = -\beta z + xy, \qquad \hat{z}' = -\beta\hat{z} + x\hat{y}.$$

The fact that synchronization occurs and that the manifold D is normally hyperbolic can be shown again by finding an appropriate quadratic Lyapunov functional [20]. This implies directly that the generalized Lyapunov number α of D is negative. The synchronization manifold $M \subset D$ can be taken as the inflowing ellipsoid of the Lorenz system [19]. k-normal hyperbolicity is somewhat harder to establish since we do not have control over the coupling as in the last example. However, Fig. 1 illustrates that synchronization as measured by the correlation between the time series of z and $\hat{z}$ does persist under some perturbations.

Local equivalence of unidirectional and bidirectional couplings.—Normal k-hyperbolicity of the synchronization manifold M implies other useful properties of the dynamics in the vicinity of M. It can be shown that each orbit γ on M is "shadowed" by orbits off of M. In particular, if we choose $p \in M$, there is an n-dimensional manifold (or fiber) $W^s(p)$, such that if $x \in W^s(p)$, then $d[\phi^t(x), \phi^t(p)] \leq Ce^{-\alpha t}$ as $t \to \infty$, where α is the generalized Lyapunov number measuring the contraction in the direction normal to M and ϕ represents the flow induced by the vector field. The manifolds $W^s(p)$ are smooth, unique, and, under certain nonresonance conditions, depend smoothly on $p \in M$.

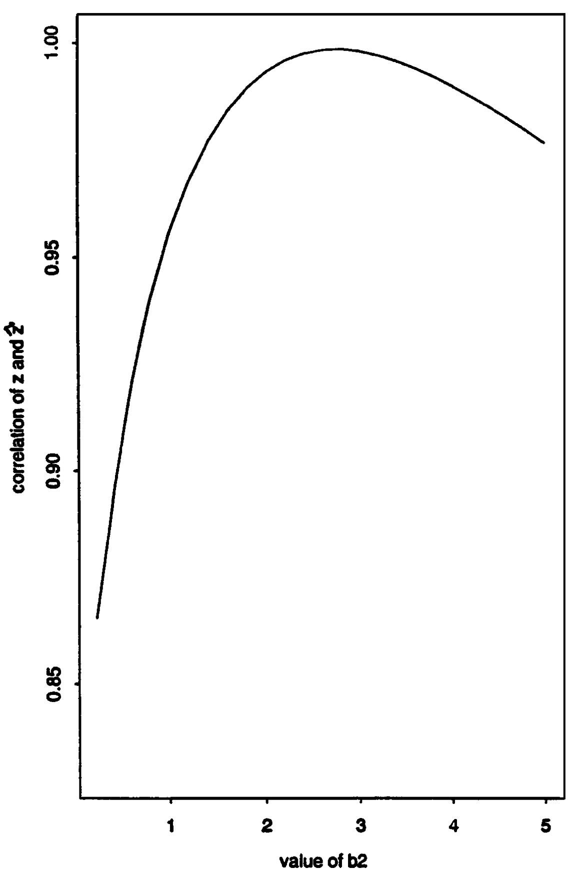

FIG. 1. The correlation between the driving system variable z and the subsystem variable $\hat{z}$ in example 2 as a function of β_2 with $\beta = \frac{8}{3}$.

Given a pair of coupled differential equations, we say that the coupling is *unidirectional* if one of the equations is independent of the other [i.e., Eqs. (1) can be written as $x' = f(x)$ and $y' = g(x,y)$]. The coupling can be unidirectional in only part of the phase space. Physically, this would mean that, in part of the phase space, the behavior of one system has no influence on the behavior of the other. An example is provided by two electrical circuits coupled through a diode. The following definition makes this notion precise:

Definition 3.—Two dynamical systems are said to be unidirectionally coupled on $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ *if* $g(x,y)$ *is independent of* x *for each* $y \in \Pi^2(\Omega)$ *or if* $f(x,y)$ *is independent of* y *for each* $x \in \Pi^1(\Omega)$.

In this situation, manifolds $W^s(p)$ which are pieces of affine spaces will be of particular importance.

Definition 4.—If there exists an open set $\Omega \subset \mathbb{R}^{2n}$ *such that* $W^s(p) = \{p\} \times \mathbb{R}^n \cap \Omega$, *then the fiber* $W^s(p)$ *is called* straight on Ω.

An example of a straight fiber is given in Fig. 2. Straight fibers are directly related to unidirectional synchronization as the statement of the following lemma shows. Notice that, in the following discussion, the systems are required to be *stably synchronized* which implies normal k-hyperbolicity of the synchronization manifold M.

Lemma 1.—The coupling between two stably synchronized dynamical systems is unidirectional in a neighborhood Ω *of the synchronization manifold* M *if the fibers* $W^s(p)$ *are straight in* Ω. Proof: Let us illustrate the proof of this lemma in two dimensions since the higher dimensional case is a direct extension of this argument. Assume that the coupling is not unidirectional in Ω. This means that there exists a point $p = (x_0, y_0) \in M$ and points (x_0, y_1), (x_0, y_2) such that the speeds in the x direction of (x_0, y_1) and (x_0, y_2) are different. If these two points below to $W^s(p)$, then $\phi^t[(x_0, y_i)] \in W^s[\phi^t(p)]$ for $i = 1, 2$. However, since the speed in the x direction is different for these two points after some small time, the flow will take them to points with different x coordinates. Therefore, for some small ϵ, the fiber $W^s[\phi^\epsilon(p)]$ cannot be straight.

Figure 2 gives a schematic representation of this situation. □

This lemma is the essential step in the proof of the following theorem:

Theorem 1.—If, for a pair of bidirectionally coupled, stably synchronized *dynamical systems, the fibers* $W^s(p)$ *depend smoothly on* $p \in M$ *and* $\{p\} \times \mathbb{R}^n$ *is not tangent to* M *for any* $p \in M$, *then there exists a smooth change of coordinates in some neighborhood* Ω *of* M *which*

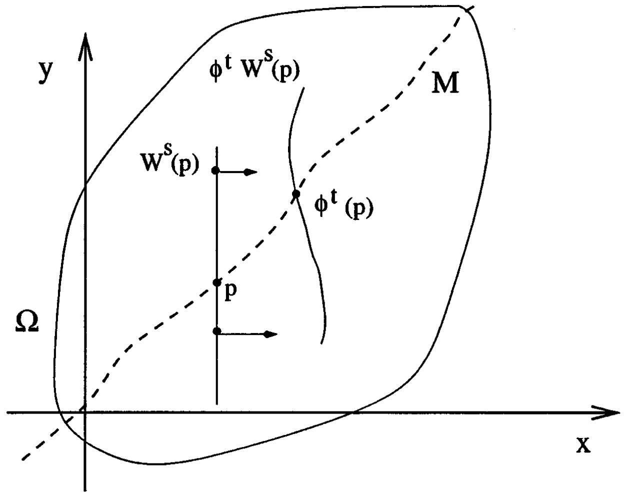

FIG. 2. A schematic representation of Lemma 1. If the velocity of the points in the fiber W^s are different, it will "bend" under the flow.

take the original bidirectionally coupled system into a unidirectionally coupled system.

Since the fibers $W^s(p)$ are smooth and by the hypothesis of the theorem depend smoothly on $p \in M$, they give a coordinate system in the neighborhood of M in a natural way. The problematic case in which $\{p\} \times \mathbb{R}^n$ is tangent to M resulting in a singularity is ruled out. The fibers $W^s(p)$ are straight in this coordinate system by definition, and so by the previous lemma the coupling is unidirectional. A rigorous proof of this assertion in a more general setting and conditions for the smooth dependence of $W^s(p)$ on p can be found in [17].

Example 3.—Consider a unidirectional coupled system $x_1' = f(x_1)$ and $x_2' = g(x_1, x_2)$. Under the coordinate transformation,

$$(y_1, y_2) = [\tfrac{1}{4}(3x_1 + x_2), -\tfrac{1}{4}(x_1 + 3x_2)]$$
$$= \xi^{-1} \circ \eta \circ \xi(x_1, x_2),$$

where $\xi(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$ rotates the space so that the diagonal D goes to $\xi_1 = 0$. In one dimension, the straight fibers go to lines that are at a 45° angle with the ξ_1 axis. The transformation $\eta(\xi_1, \xi_2) = (\xi_1, 2\xi_2)$ tilts the fibers so that they are not straight any longer after we go back by ξ^{-1}. The inverse of this linear change of coordinates will then take a bidirectionally coupled system to a unidirectionally coupled system.

In general, it is not possible to obtain an explicit description of the manifolds $W^s(p)$ of a bidirectionally coupled system; however, this theorem shows that, under certain conditions, it is sufficient to study unidirectionally coupled systems to obtain information about bidirectionally coupled systems.

In conclusion, this paper gave an outline of a simple setting in which synchronization can be studied and many generalizations are possible: (i) Stable synchronization of n coupled oscillators can be defined in terms of a k-hyperbolic invariant manifold in the phase space of the entire system. (ii) By relaxing the requirement that the synchronization manifold be diagonal-like, we can use this approach to study generalized synchronization. (iii) Other changes in definition 1 could be made to include synchronization between systems with different degrees of freedom.

The well developed theory of invariant manifolds can then be used to understand these phenomena and to prove properties of the system which are not obvious outside of this framework.

I thank Eugene Wayne and Idel Bronshteyn for their help in the preparation of this paper.

*Present address: Department of Mathematics, Boston University, 111 Cummington Street, Boston, MA 02215.

[1] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Radiophys. Quantum Electron. Engl. Transl. **29**, 747–751 (1986).
[2] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32–47 (1983).
[3] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821–824 (1990).
[4] C. Tresser, P. Worfolk, and H. Bass, Chaos **5**, 693–699 (1995).
[5] J. K. Hale, J. Dyn. Differ. Equ. **9**, 1–52 (1997).
[6] T. Ushio, Phys. Lett. A **198**, 14–22 (1995).
[7] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804–1807 (1996).
[8] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816–1819 (1996).
[9] H. D. I. Abarbanel, N. F. Rulkov, and M. Sushchik, Phys. Rev. E **53**, 4528–4535 (1996).
[10] L. O. Chua and M. Itoh, J. Circuits Syst. Comput. **3**, 93–108 (1993).
[11] R. Roy and K. S. Thornburg, Jr. (to be published).
[12] R. E. Mirolla and S. H. Strogatz, SIAM J. Appl. Math. **50**, 1645–1662 (1990).
[13] S. Watanabe, H. S. J. van der Zant, S. H. Strogatz, and T. P. Orlando (to be published).
[14] K. Pyragas, Phys. Lett. A **181**, 203–210 (1993).
[15] U. Parlitz *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 973–977 (1992).
[16] R. Brown and N. Rulkov (to be published).
[17] I. U. Bronstein and A. Ya. Kopanskii, *Smooth Invariant Manifolds and Normal Forms* (World Scientific, Singapore, 1994).
[18] S. Wiggins, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems* (Springer-Verlag, New York, 1994).
[19] C. Sparrow, *The Lorenz Equations, Bifurcations, Chaos, and Strange Attractors* (Springer-Verlag, New York, 1982).
[20] R. He and P. G. Vaidya, Phys. Rev. A **46**, 7387–7392 (1992).