The Mathematics of Musical Instruments

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Abstract

This article highlights several applications of mathematics to the design of musical instruments. In particular, we consider the physical properties of a Norwegian folk instrument called the willow flute. The willow flute relies on harmonics, rather than finger holes, to produce a scale which is related to a major scale. The pitches correspond to fundamental solutions of the one-dimensional wave equation. This "natural" scale is the jumping-off point for a discussion of several systems of scale construction—just, Pythagorean, and equal temperament—which have connections to number theory and dynamical systems and are crucial in the design of keyboard instruments. The willow flute example also provides a nice introduction to the spectral theory of partial differential equations, which explains the differences between the sounds of wind or stringed instruments and drums.

1 Introduction

The history of musical instruments goes back tens of thousands of years. Fragments of bone flutes and whistles have been found at Neanderthal sites. Recently, a $9,000$-year-old flute found in China was shown to be the world's oldest playable instrument.[1] These early instruments show that humans have long been concerned with producing *pitched sound*—that is, sound containing predominantly a single frequency. Indeed, finger holes on the flutes indicate that these prehistoric musicians had some concept of a musical scale.

The study of the mathematics of musical instruments dates back at least to the Pythagoreans, who discovered that certain combinations of pitches which they considered pleasing corresponded to simple ratios of frequencies such as 2:1 and 3:2. The problems of tuning, temperament and acoustics have since occupied some the brightest minds in the natural sciences. Marin Marsenne's treatise on tuning and acoustics *Harmonie Universelle* (1636) [19], H. v. Helmholtz's *On the Sensations of Tone* (1870) [15], and Lord Rayleigh's seminal *The Theory of Sound* (1877) [21] are just three outstanding examples.

Many pages have been written on this subject. We mean to present an overview and let the interested reader find more detailed discussions in the references, and on our web site www.sju.edu/∼rhall/newton.

2 The willow flute

In this section we consider the physical properties of a Norwegian folk flute called the *seljefløyte*, or willow flute. This instrument can be considered "primitive" in that it does not rely on finger holes to produce different pitches. Rather, by varying the strength with which he or she blows into

[1] Pictures and a recording of this flute are available at http://www.bnl.gov/bnlweb/flutes.html.

Figure 1: Musicologist Ola Kai Ledang playing the willow flute

the flute, the player selects from a series of pitches called *harmonics* whose frequencies are integer multiples of the flute's lowest tone, called the *fundamental*. The willow flute's scale is approximately a major scale with a sharp fourth and flat sixth, and plus a flat seventh.

The willow flute is a member of the recorder family, though it is held transversally. The flute is constructed from a hollow willow branch (or, more recently, a PVC pipe[2]). One end is open and the other contains a slot into which the player blows, forcing air across a notch in the body of the flute. The resulting vibration creates standing waves inside the instrument whose frequency determines the pitch. The recorder has finger holes which allow the player to change the frequency of the standing waves, but the willow flute has no finger holes. However, it is evident from the tune *Willow Dance* (Figure 2), as performed by Hans Brimi on the willow flute [8], that quite a number of different tones can be produced on the willow flute. How is this possible?

Figure 2: Transcription of *Willow Dance*, as performed by Hans Brimi

The answer lies in the mathematics of sound waves. Let u be the pressure in the tube, x be the position along the length of the tube, and t be time. Since the pressure across the tube is close to constant, we can neglect that direction. We will choose units such that the pressure outside the tube is 0.

The one-dimensional wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

provides a good model of the behavior of air molecules in the tube, where a is a positive constant. Since both ends of the tube are open, the pressure at the ends is the same as the outside pressure.

[2]Instructions for making a PVC "willow" flute and sound clips may be found at the quirky but informative web site http://www.geocities.com/SoHo/Museum/4915/SALLOW.HTM.

That is, if L is the length of the tube, $u(0,t)=0$ and $u(L,t)=0$. Solutions to the wave equation are sums of solutions of the form

$$u(x,t)=\sin\frac{n\pi x}{L}\left(b\sin\frac{an\pi t}{L}+c\cos\frac{an\pi t}{L}\right)$$

where $n=1,2,3,\ldots$, and b and c are constants. The derivation of this solution may be found in most textbooks on differential equations such as [7].

How does our solution predict the possible frequencies of tones produced by the flute? For now, let's just consider solutions which contain one value of n. Fix n and x and vary t. The pressure, u, varies periodically with period $\frac{2L}{an}$. Therefore,

$$\text{frequency}=\frac{an}{2L}$$

for $n=1,2,3,\ldots$.

This formula suggests that there are two ways to play a wind instrument: either change the length L, or change n.[3] Varying L continuously, as in the slide trombone or slide whistle, produces continuous changes in pitch. The more common way to change L is to make holes in the tube, which allow for discrete changes in pitch. The other way to vary the pitch is to change n—that is, to jump between solutions of the wave equation. The discrete set of pitches produced by varying n are the harmonics. Specifically, the pitch with frequency $\frac{an}{2L}$ is called the nth harmonic; if $n=1$ the pitch is the fundamental or first harmonic.[4]

The sequence of ratios of the frequency of the fundamental to the successive harmonics is 1:1, 1:2, 1:3, 1:4, ... (note the connection to the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$). If the first harmonic is a C, then the next five harmonics are C′, G′, C″, E″, and G″, where the prime denotes the pitch an octave higher. The fourth, fifth, and sixth harmonics form what is called a *major chord*, one of the primary building blocks of Western music. However, this solution still doesn't completely explain the willow flute.

Let's take a closer look at the willow flute player's right hand (Figure 1). The position of his fingers allows him to cover or uncover the hole at the end of the flute. In other words, the player is able to change the boundary condition at that end. At the closed end, air pressure is constant in the x direction, so the boundary conditions become $u(0,t)=0$ and $u_x(L,t)=0$. Solving the wave equation as before, we get a set of solutions for which

$$\text{frequency}=\frac{an}{4L}$$

for $n=1,3,5,7,\ldots$. Since the original value of frequency was $\frac{an}{2L}$, closing the end has dropped the fundamental an octave and restricted the harmonics to odd multiples of the fundamental frequency. Combining the harmonics produced with end closed and with end open, we see that in the third octave (relative to the fundamental of the open pipe) there is a nine-note scale available, which we will call the flute's *playing scale*. As an aid to visualization, these two sets of harmonics are shown in their approximate positions on a piano keyboard in Figure 3, assuming the fundamental of the open pipe is tuned to a C. However, it should be emphasized that pianos are not commonly tuned to the willow flute's scale!

How necessary is this second set of harmonics? The harmonics produced by the open pipe in the fourth octave from its fundamental form the same scale as the combined open- and closed-end

[3] For stringed instruments, which are also governed by the one-dimensional wave equation, there are three ways, since you can also change the value of a by changing the tension on the string (for instance, by string bending) or by substituting a string of different density.

[4] The term *overtone* is also used to describe these pitches, but the nth harmonic is called the $(n-1)$st overtone.

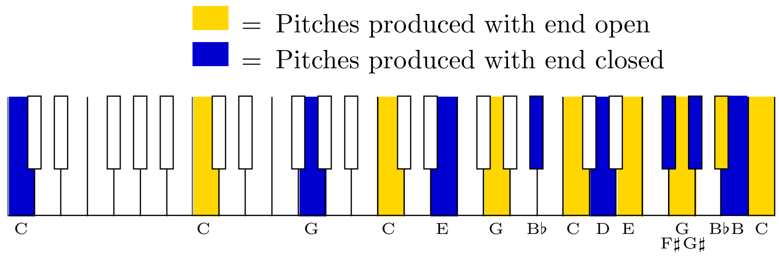

Figure 3: Approximate location of the willow flute's pitches on a piano

scale, but an octave higher. In fact, if we don't care which octave we're in, the willow flute can theoretically produce pitches arbitrarily close to *any* degree of the scale, even without changing the boundary condition. However, this solution is impractical. The higher harmonics are not only less pleasing to the ear, but also more difficult to control. In order to produce the rapid note changes required for the *Willow Dance*, the player needs the second set of harmonics. Some willow flute players extend this technique by covering the end hole only halfway to produce an intermediate set of pitches, or by continuously changing the boundary condition to produce a continuous change in pitch.

So far, we have only considered those solutions to the wave equation of the form

$$u(x,t) = \sin\frac{n\pi x}{L}\left(b\sin\frac{an\pi t}{L} + c\cos\frac{an\pi t}{L}\right)$$

A more general solution is a sum of several of these. In such a sum, the terms containing the smallest values of n generally have the greatest amplitude and determine the pitch and character of the sound. In particular, the fundamental predominates and it is perceived as the pitch of the sound. The relative volumes of the harmonics explains how we can distinguish the sounds of different musical instruments. For instance, the clarinet's sound contains only odd harmonics, as does the sound of the willow flute with the end closed. Sethares' fascinating book [25] proposes that Western music uses scales based on small integer ratios of frequencies precisely because the sound of winds and strings consists of harmonics. When two such instruments play notes from the same scale, many of the harmonics produced by the instruments will correspond, creating an effect pleasing to the ear.

To fully describe the acoustic properties of instruments, it is also necessary to take into account nonlinear effects. This is still a very active area of research, and a good overview may be found in [12].

3 From melody to harmony: keyboard instruments

In this section, we use the willow flute as the jumping-off point for a discussion of scale construction. The willow flute's playing scale is appealing mathematically in that each ratio of frequencies within the scale can be expressed with small integers. And, since the music traditionally played on the willow flute is exclusively melodic and centered on the key of its fundamental, we are less concerned with the relationships of the notes within its scale to one another. However, when we try to use this system to design keyboard instruments, problems arise. For instance, we would like the 4:5:6 relationship of the major chord to be replicated in several locations on our keyboard, and we would like our instrument to sound "good" in several different keys. It turns out that these goals are not

note	1	2	3		4	5	6		7	8	9
willow flute	1:1	9:8	5:4		11:8	3:2	13:8		7:4	15:8	2:1
just intonation	1:1	9:8	5:4	4:3		3:2		5:3		15:8	2:1
note	1	2	3	4		5		6		7	8

Table 1: Comparison of just intonation and the willow flute's playing scale

simultaneously achievable. The story of the attempts to resolve this issue illustrates one of the most interesting intersections of mathematics and aesthetics.

3.1 Just intonation

We begin by writing down the ratios of frequencies of the nine notes in the willow flute's playing scale to the first note of that scale (Table 1). These are computed by finding the relationship of each note to the flute's first harmonic and then dividing to find their relationship to each other. Observe that the sequence of ratios in this scale can be written 8:8, 9:8, 10:8, ...

A major chord is comprised of three notes whose ratio of frequencies (give or take an octave) is 4:5:6. We see that there are two major chords in the willow flute's scale: the chord formed by the first, third, and fifth notes in its scale (called the I chord), and the chord formed by the fifth, eighth, and second notes (called the V chord). *Just intonation* is based on an eight-note scale which may be decomposed into three major chords: I, V, and IV, which contains the fourth, sixth, and eighth notes of the just intonation scale. Many versions of just intonation were proposed between the 15th and the 18th century, most differing on how to construct the remaining notes in the chromatic scale [3], [5]. A history of the various systems and practical guidelines on how they may be implemented, as well as discussion of Mersenne's work on this problem, is found in [16].

Just intonation has several problems. One of the most glaring is the ratio of the sixth to second degrees of the scale, which is 40:27, rather than 3:2. When just intonation is used, the same note may have a different pitch in several keys. For instance, the ratio of A to G is equal to 10:9 when we're playing in C, rather than 9:8 when we're playing in G. The players of stringed and wind instruments can make these small adjustments in pitch as they play; however, a different system must be devised for instruments with fixed pitch. Various compromises have been proposed, including *tempered* scales which involve adjustments to just intonation. Another solution for keyboard instruments is to add keys, or allow for the alteration of pitch by an application of levers or pedals. Many ingenious methods were developed to translate these ideas into practice, from the earliest known being the organ of St. Martin's at Lucca having separate keys for E♭ and D♯ up to the "Enharmonium" of Tanaka which separated the octave into 312 notes [3]. One of the few instruments of this type in use today is the English concertina, which has separate buttons for E♭ and D♯ and for A♭ and G♯. Most modern concertina players opt to have their instruments tuned to equal temperament, however.

3.2 The Pythagorean scale

In the previous construction we saw that intervals are formed by multiplying a fundamental frequency by a rational number. Pythagoras discovered that the 2:1 ratio of an octave and the 3:2 ratio of a fifth are particularly consonant and used them as the basis for a scale. His construction avoids the problem of some fifths being out of tune in just intonation. The idea is to start with a fundamental frequency and multiply repeatedly by $\frac{3}{2}$ to obtain other notes in the scale. Two notes that are an octave apart represent the same degree of the scale. Therefore, if multiplying a frequency f by $\frac{3}{2}$

gives us a frequency which is not in the octave in which we started—that is, if $\frac{3}{2} \times f > 2$—we can divide the result by 2 to return to the original octave.

It will be convenient to work with logarithms of base 2 of a given frequency, rather than the frequency itself. If we choose units such that middle C has frequency 1, then in the logarithmic units middle C has frequency $\log_2 1 = 0$, while C′, an octave above middle C, has logarithmic frequency 1 since $2^1 = 2$. Setting $x = \log_2 f$ and taking the logarithm base 2 we obtain

$$x \to x + \log_2 \frac{3}{2}$$

as the mapping taking a tone to its fifth in logarithmic units. Since dividing by 2 corresponds to subtracting 1 on the logarithmic scale, and since we subtract 1 only if $x + \log_2 \frac{3}{2} > 1$, we obtain the following map on the interval $[0, 1]$:

$$x \to x + \log_2 \frac{3}{2} \pmod 1.$$

By identifying the endpoints of the interval $[0, 1]$, this map can be thought of as an irrational rotation of a circle. It is a well-known fact that an initial point x_0 never returns to itself under the iteration of such a map. Rather, its images fill the circle densely [11]. Therefore, if we move by fifths, we will never return to the the frequency with which we started. This fact has unfortunate consequences for the construction of a scale, as was discovered by the Pythagoreans. This problem is most apparent in instruments with fixed pitch.

3.3 Equal temperament

Equal temperament involves approximating an irrational rotation of the circle by a rational one. There is a natural geometric way to think about this approximation. The graph of the line $y = \mu x$ intersects the vertical lines $x = q$, where q is a positive integer, at the points μq. The decimal part of this number is exactly the q-th iterate of 0 under the rotation map $x \to x + \mu \pmod 1$.

If μ is irrational this line does not pass through any points of the lattice $\mathbf{Z} \times \mathbf{Z}$, and therefore an irrational rotation of the circle has no periodic orbits. A line passing through a point $(q, p) \in \mathbf{Z} \times \mathbf{Z}$ that lies close to $y = \mu x$ gives rise to a rotation $x \to x + \frac{p}{q} \pmod 1$ which approximates the rotation $x \to x + \mu \pmod 1$. If the fraction $\frac{p}{q}$ in the approximate mapping is in reduced form, the orbit of any point has period q, and the points of the orbit are distributed uniformly around the circle. In other words, we can use this approximation to divide the octave into q equal parts. Scales constructed in this way are called *equally tempered*.

The following is a geometric way to find a sequence of points $(q, p) \in \mathbf{Z} \times \mathbf{Z}$ which are successively closer to $y = \mu x$.[5] Imagine a string attached to infinity extending to the origin along the line $y = \mu x$. Also imagine that a nail is driven through each point in the plane with positive integer coordinates. If we pull the free end of the string up or down it will touch the nails which are closest to the line $y = \mu x$. The region bounded by the pulled-up string and the line $x = 1$ is the convex hull of the points above $y = \mu x$. The pulled-down string bounds the convex hull of the points below $y = \mu x$.

For instance, if $\mu = \log_2 \frac{3}{2}$, the string touches $(1, 1), (5, 3), (41, 24), (306, 179), \ldots$ when it is pulled up, and $(2, 1), (12, 7), (53, 31), \ldots$ when it is pulled down, as shown in Figure 4, where the string bounds the gray areas. Therefore $\log_2 \frac{3}{2}$ is approximated well by the sequence $1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \ldots$.

The meaning of "approximated well" can be made precise. Consider the following construction: Let $e_{-1} = (0, 1)$ and $e_0 = (1, 0)$. If e_{k-1} and e_k are given, let e_{k+1} be the vector obtained by adding e_k to e_{k-1} as many times as possible without crossing $y = \mu x$.

[5] This discussion follows the presentation in [1]. The construction is originally due to F. Klein.

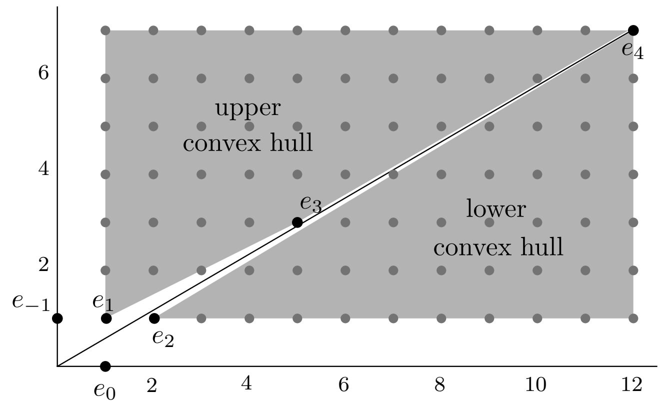

Figure 4: Approximation of the line $y = (\log_2 \frac{3}{2})x$

Proposition 1 *The oriented area of the parallelogram spanned by the vectors e_{k-1} and e_k is $(-1)^k$, when orientation is taken into account.*

Proof. Every subsequent parallelogram shares a side and altitude with its predecessor. □

Corollary 1 *The points e_k, $k > 0$, are extreme points of either the upper or lower convex hulls.*

Proof. If the points e_{k-1} and e_k were not on the convex hull, the parallelogram formed by the vectors e_{k-1} and e_k would have to contain a point in $\mathbf{Z} \times \mathbf{Z}$. By Pick's Theorem [26] such a parallelogram has area greater than 1 contradicting Proposition 1. □

The following is another straightforward corollary [1].

Corollary 2 *If q_k and p_k are the coordinates of e_k, $k > 0$, then*

$$\left| \mu - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

This corollary shows that the numbers we obtained in the geometric constructions are the convergents obtained in the continued fraction expansion of μ, and in this sense the best rational approximations to μ. Detailed discussions of continued fractions can be found in [14], while their applications to music are discussed in [2]. In particular, note that $\log_2 \frac{3}{2}$ is a transcendental number, which means that the complexity of its rational approximations increases rapidly.

There are several things to consider when choosing any of these approximations as the basis for a scale. The period of a rational rotation is determined by the denominator of the fraction $\frac{p}{q}$. A large denominator leads to a scale with many notes. This is impractical, due to the physical constraints of instruments and our inability to distinguish tones which are very close in pitch.

It is at least in part due to these considerations that the approximation $\log_2 \frac{2}{3} \approx \frac{7}{12}$ is used as the basis of Western music. Figure 3.3 shows how the evenly spaced tones obtained from this approximation divide the octave into 12 and 41 parts (41 parts providing the next best approximation). The gray dots represent just intonation. It is evident that the just intonation scale is fairly well approximated by the equally tempered twelve-note scale. This is somewhat fortuitous, as we have made no effort to approximate the 5:4 ratio of a third in the above construction. Whether the benefits obtained by this construction outweigh the price that we have to pay in having all intervals

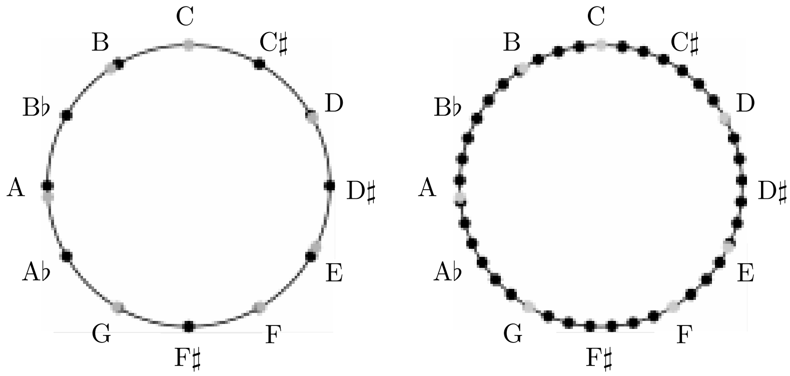

Figure 5: The circle divided into 12 and 41 equal parts

"impure" is still a subject of debate. A *Mathematica* application which explores this construction in more detail is available at www.sju.edu/∼rhall/newton.

There are other ways of constructing equally tempered scales. We could try to find rational numbers with equal denominators that approximate both the fifth and third well. The associated rotation would divide the circle into a number of equal parts. This approach leads to the theory of higher-dimensional continued fractions which are still a subject of much research [2], [18]. Many equally-tempered scales were explored in the past, from the 17-note Arabian scale to the 87-note division praised by Bosanquet. Easley Blackwood [6] has written compositions for each of the equally tempered scales containing 13 to 24 tones. J. M. Barbour and D. Benson present excellent historical reviews of this subject [3], [5]. The reader is invited to compare the merits of the different subdivisions using the *Mathematica* program available on the authors' web page.

The scale of twelve equally tempered notes leads to another interesting question. The distance between the frets of an equally tempered stringed instrument such as a guitar or a lute has to be scaled by the ratio $2^{\frac{1}{12}} : 1$. Since $2^{\frac{1}{12}} = (2^{\frac{1}{3}})^{\frac{1}{4}}$ this problem is equivalent to duplicating a cube, a task which cannot be accomplished by Euclidean methods. Constructing this ratio with the methods of measurement available in the 16th and 17th century was a difficult task and finding an approximation was of considerable utility. A number of interesting approaches, including ingenious constructions by Galileo Galilei's father and Stähle are discussed in [4].

4 Drums and other higher-dimensional instruments

So far we have considered only instruments that are essentially one-dimensional. All stringed and wind instruments fall into this category. Percussion instruments such as drums and bells do not. Why is that? Let us think of a drum with a circular drumhead as a circular domain of radius R around the origin in $\mathbf{R}^2$ obeying the wave equation with fixed boundary. Using polar coordinates (r, ϕ) and separation of variables it can be shown that the transversal displacement of the drumhead at time t is given by $F(r, \phi, t) = g(t)\, f_1(r)\, f_2(\phi)$ where

$$g''(t) + c^2\lambda g(t) = 0 \qquad f_2''(\phi) + \mu f_2(\phi) = 0 \tag{1}$$

$$f_1''(r) + \frac{1}{r} f_1'(r) + (\lambda - \frac{\mu}{r^2}) f_2(r) = 0. \tag{2}$$

The constant c is related to the physical properties of the material, and λ and μ are determined from the conditions $f_2(-\pi) = f_2(\pi)$ and $f_1(R) = 0$. See [20] for more details.

The equations for g and f_2 are easy to solve. The constraints on f_2 force $\mu = m^2$ which means that (2) is exactly the m-th Bessel equation whose solution are given in terms of the m-th Bessel function as $f_1(r) = J_m(r\sqrt{\lambda})$. Since the drumhead is fixed along its boundary this means that $f_1(R) = J_m(R\sqrt{\lambda}) = 0$ and so λ can only assume the values

$$\lambda_n = \left(\frac{x_n^{(m)}}{R}\right)^2 \tag{3}$$

where $x_n^{(m)}$ are the zeros of the m-th Bessel equation. The different values of λ determine the frequencies of oscillation of the different modes, as in the case of the willow flute. Since the zeros are irrationally related, it follows that the frequencies of oscillations of the drumhead cannot be rational multiples of each other. This is why drums using a freely oscillating circular membrane produce notes of a discernibly different tonal character than that of one-dimensional instruments.[6]

Of course, we do not have to restrict ourselves to the case of circular drums. We can consider the wave equation on a general domain D in $\mathbf{R}^2$ and look for solutions that satisfy $F(x, y, t) = 0$ on the boundary ∂D. Separating variables as $F(x, y, t) = \Psi(t)\Phi(x, y)$ lets one conclude that the general solution is of the form $F(x, y, t) = \sin(\sqrt{\lambda}t)\Phi(x, y)$ where

$$\nabla^2\Phi + \lambda\Phi = 0 \text{ in } D \qquad \text{and} \qquad \Phi = 0 \text{ on } \partial D \tag{4}$$

As we have seen before, a solution to this problem exists only for certain values of λ known as eigenvalues. These eigenvalues depend on the shape of the drum D, and are the squares of the frequencies of vibrations of the different modes.

In his beautiful article *Can one hear the shape of a drum?* M. Kac asked whether two drums with the same eigenvalue spectrum necessarily have the same shape [17]. Kac proved that certain characteristics of the domain, such as its area and circumference, are indeed determined by the eigenvalues. The general problem remained unsolved for 24 years until Gordon et. al. showed that two non-congruent drums can have the same spectrum [13]. For an explicit construction of two such drums see [9].

We still have one dimension remaining: can we characterize the sound of three-dimensional instruments? All three-dimensional instruments fall into the class of percussion instruments. If we write the wave equation for some simple geometric shapes—for instance, a rod—we can conclude that the frequencies of the different modes of vibration are not rationally related.[7] There are, however, three-dimensional instruments whose sounds are similar to the sounds of one-dimensional instruments: the marimba, the glockenspiel, claves, and others. There are several ways in which this is achieved. Some three-dimensional objects, such as rods, vibrate predominantly in a one-dimensional fashion. On the other hand, the bars on a marimba corresponding to lower tones have deep arches on one side. These are cut in such a way that the first two modes of vibration of the

[6]One instrument that does not fit within this picture is the timpanum. The membrane of the timpanum is truly two-dimensional, and yet its sound is similar to that of one-dimensional instruments. Unlike a tambourine, the timpanum has a closed bottom and its vibrations change the pressure in the cavity beneath the oscillating membrane. Therefore its membrane is not freely oscillating and additional nonlinear forcing terms have to be added to the wave equation to accurately describe its behavior. By carefully tuning the bowl beneath the membrane, the frequencies of the first few modes of vibration can be related as 2:3:4:5 [10].

[7]Instruments of this type are uncommon in Western music. Sethares [25] shows that the scales used by the Indonesian gamelan are related to the gamelan instruments' spectra, which consist of tones which are not rationally related to the fundamental.

bar are rationally related. Since the higher notes are above the 2000 Hz range, they are not as important in determining the perceived sound of those bars. Excellent descriptions of percussion instruments can be found in the works of Rossing [22], [23], [24] (a more popular treatment), and [12] (with Fletcher).

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