An Example Regarding Chaotic Phase Synchronization

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The following is an example which illustrates the phenomenon discussed in Section 7 of "A Geometric Theory of Chaotic Phase Synchronization."

We consider a system with a three dimensional return map, so that $x = (y, z, w) \in [0, 1] \times [0.2, 0.8] \times [0, 1]$ in the following equation

$$\begin{aligned} \rho_{n+1} &= \rho_n + \eta(x_n) + \epsilon\Gamma(\rho_n, x_n) \\ x_{n+1} &= F(x_n) \end{aligned} \tag{1}$$

The dynamics on the Poincaré section are defined by

$$\begin{aligned} y_{n+1} &= 4y_n(1 - y_n) \\ z_{n+1} &= z_n + \delta(w_n - 1/2) \\ w_{n+1} &= -0.05(2.5w_n - 1.25)^3 + 1.125w_n - 0.0125 - 0.1z_n \end{aligned} \tag{2}$$

with $\delta = .01$. The (z, w) subsystem is a discrete version of the van der Pol oscillator and it has essentially been analyzed in [?]. Using a quasi-static approach it is straightforward to show that for sufficiently small values of δ, the w-coordinate of a typical orbit spends long times close to 0 and 1, and that it transitions rapidly between these two states. The length of the time intervals during which w is close to 0 or 1 can be made arbitrarily long by decreasing δ.

The dynamics of y is governed by a logistic map on the interval $[0, 1]$. We assume that there are only two elements in the partition of $\Sigma = [0, 1] \times [0.2, 0.8] \times [0, 1]$ defined as $P_1 = \{(y, z, w) \in \Sigma | y \geq w\}$ and $P_2 = \{(y, z, w) \in \Sigma | y < w\}$. Since the w-coordinate of an orbit of system (2) spends long times in the vicinity of the

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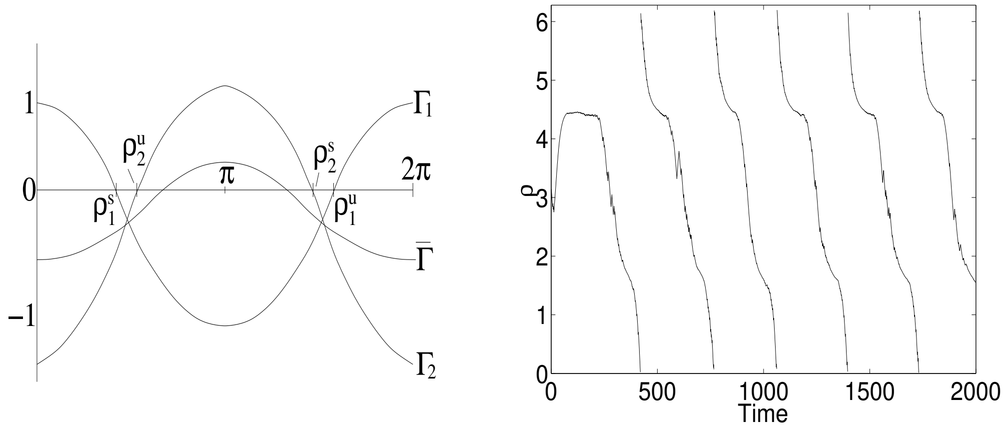

Figure 1: On the left are graphs of possible Γ functions whose average, $\bar{\Gamma}$, has a fixed point. However, because the convergence to the average is not sufficiently fast, the system will not phase lock. The evolution of ρ for such a system is shown on the right.

points $w = 0$ and $w = 1$, and the invariant measure of the logistic map is symmetric about $y = 1/2$, there are long subsequences in typical orbits of the Poincaré map that remain mostly in one of the partition elements. The typical length of such subsequences can be made arbitrarily long by decreasing δ.

Now consider a system for which $\Gamma_1 = \cos(\rho_n)$ and $\Gamma_2 = c + \cos(\rho_n + \theta)$ where θ and c are chosen so that we obtain the two graphs in Fig. 1. If we choose $\rho_0 = \pi$ as an initial value and $x_n \in P_1$ for a sufficiently long time, then the iterates ρ_n will approach the fixed point ρ_1^s of Γ_1, from the right. Eventually, the sequence of iterates ρ_n will cross the unstable fixed point ρ_2^u of Γ_2. If x_n switches to P_2 after this occurs, the sequence ρ_n will continue moving to the left and approach ρ_2^s. This process can repeat as long as the time spent in P_1 and P_2 is sufficient for ρ to cross ρ_2^u and ρ_1^u respectively. As shown in Fig. 1, ρ will decrease through the entire interval $[0, 2\pi]$ even though the average $\bar{\Gamma}$ has a stable fixed point. This illustrates the process referred to as slipping.