Synchronization of Chaotic Systems and Invariant Manifolds

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Abstract

Despite the extensive research in chaotic synchronization, a mathematical framework within which the phenomenon could be described and analyzed has not been fully developed. In the present work the theory of normally hyperbolic invariant manifolds is used to create such a framework. This approach allows us to analyze in a unified way synchronization schemes that have been treated distinctly in the literature. A rigorous analysis of techniques and quantities used to predict and measure chaotic synchronization is given, and some cautionary examples are presented.

1 Introduction

The study of synchronization of chaotic systems has its origins in the work of Fujisaka and Yamada [2], Afraimovich, Verichev, and Rabinovich [1], and Pecora and Carroll [4]. The applications of chaotic synchronization in secure communications, which were demonstrated experimentally in 1991, engaged the interest of the physics and engineering communities, and it has been an active area of research since. Many different aspects and incarnations of chaotic synchronization were studied: system-subsystem synchronization [4],[20], synchronization in unidirectionally and bidirectionally coupled systems [9],[10], anti-phase synchronization [10],[11], partial synchronization [3] and generalized synchronization [7],[8]. The phenomenon has been observed in many physical systems, for instance mechanical and electrical systems [1],[12], laser systems [13], biological systems [14] and Josephson junctions [15]. Chaotic synchronization has also found a number of applications in control theory [19], parameter estimation from time series [17], and the stabilization of unstable periodic orbits in chaotic attractors [18]. A good review of developments in the experimental study of synchronized chaotic systems is given in [21].

Despite the active interest, until recently a satisfactory comprehensive mathematical framework within which the various aspects of chaotic synchronization could be discussed and examined was lacking. Different aspects of the problem were studied by a number of mathematicians. Afraimovich was the first to develop analytic tools to study chaotic synchronization in lattices of diffusively coupled oscillators, and this work was continued in collaboration with Hale and Chow [25], [26], [9]. Similar work was done by Pecora, Carroll and Heagy [32], with more emphasis on numerical techniques. Tresser, Worfolk and Bass proposed a framework for system-subsystem synchronization in [31], while the problem of generalized synchronization was addressed by Hunt, Grebogi and Ott [54]. Ashwin, Buescu and Stewart studied the bifurcations by which synchronization between systems may break down, and relations with riddled basins of attraction [45], [46].

Our goal is to develop a framework within which many types of synchronization can be discussed in terms of r-hyperbolic invariant manifolds. For synchronization to be physically observable, it needs to persist under small perturbations. The present description will lead to necessary and sufficient conditions for such persistence and will illuminate relations between what appear to be different types of synchronization mechanisms. Although these ideas are implicit in many of the works already mentioned, they were first been carefully examined in [40] and [41].

In section 2 we present the basic definitions which will be used throughout the paper. In sections 3 and 4 the relations between generalized Lyapunov numbers which are used to describe the stability of normally hyperbolic invariant manifolds, and similar quantities appearing in the literature are discussed. Extensions to more general types of synchronization are provided in section 5. Different techniques for proving the existence of synchronization are provided in sections 6 and 7. Sections 9 and 10 contain a discussion of these results and point out some shortcomings of the theory.

The definitions and examples presented here will mostly concern flows in $\mathbb{R}^n$. The case of discrete dynamical systems or dynamical systems on more general manifolds involves only minor modifications.

2 Motivation and Basic Definitions

In this section we present the basic definitions of synchronization which will be used in the remainder of the article.

In a general sense synchronization between two coupled systems

$$x_1' = f(x_1, x_2) \quad x_2' = g(x_1, x_2) \quad f, g : \mathbb{R}^{2n} \to \mathbb{R}^n \tag{1}$$

means that there exists a smooth and invertible map ϕ which carries trajectories on the attractor of the first system to trajectories on the attractor of the second. This map should have the property that if an orbit of the first system approaches a trajectory $x_1(t)$ on the attractor of the first system, then the corresponding orbit of the second system approaches the trajectory $x_2(t) = \phi(x_1(t))$. Thus, once transients have died away, a knowledge of the state of the first system allows one to predict the state of the second. The existence of such a map also allows us to still speak of the attractors of each system without ambiguity.

From a geometric viewpoint we require that there exists an invariant manifold M in the phase space of network (1) which is a graph of a diffeomorphism $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Such an n-manifold with boundary will be referred to as "*diagonal-like*".

For synchronization to be observable and of physical relevance we also require M to be compact and locally attracting.

Definition 2.1 *The systems x_1 and x_2 in (1) are* synchronized *if there exists a compact, diagonal-like, smooth n-manifold M with boundary which is invariant under the flow, inflowing and locally attracting. M will be called a* synchronization manifold.

Remark 2.2 When $M = \{(x_1, x_2) \in \mathbb{R}^{2n} : x_1 = x_2\}$ the synchronization is identical. All other situations are referred to as *generalized synchronization* [7], [8].

For synchronization to be of physical interest M needs to persist under small perturbations both the coupling and the component systems. A necessary and sufficient condition for the persistence of an invariant manifold is the property that the contraction in directions normal to the manifold is r time exponentially stronger than the contraction within the manifold. Invariant manifolds with this property are called *r-normally hyperbolic* [34], [35], [37].

Definition 2.3 *The synchronization of x_1 and x_2 is called* stable *if the synchronization manifold M is r-normally hyperbolic for some $r \geq 1$.*

Notice that the qualitative behavior of each of the coupled systems may change under small perturbations, in particular if the parameters of either are close to a bifurcation value. If the synchronization manifold persists, the two systems will remain synchronized.

Remark 2.4 This definition of stable synchronization is close to the one given in [40]. The main difference is that the present definition does not require an invariant splitting of the tangent space and is thus better suited for practical applications. Moreover, the requirement that the invariant manifold is compact and inflowing is made explicit in our consideration. The case of noncompact synchronization manifolds is more subtle and will not be addressed here.

Next we give a precise definition of r-normal hyperbolicity and deal with a technical difficulty that arises when trying to apply standard theorems for the persistence of invariant manifolds to the present situation.

Let X be a C^1 vector field on $\mathbb{R}^n$ with flow F^t and let M be a compact, connected manifold with boundary which is overflowing and invariant under the flow F^t. Consider the usual metric on $\mathbb{R}^n$ so that the tangent space of M decomposes as $T\mathbb{R}^n|_M = TM \oplus N$, where N is a normal bundle of M. Let $\pi : T\mathbb{R}^n|_M \to N$ be the orthogonal projection. Define the linearization A^t of the flow parallel to the manifold, and B^t normal to the manifold as

$$\begin{array}{ll} A^t(m) = D(F^{-t}|_M)(m) & A^t(m) : T_m M \to T_{F^{-t}} M \\ B^t(m) = \pi D F^t(F^{-t}(m)) & B^t(m) : N_{F^{-t}(m)} \to N_m \end{array}$$

The generalized Lyapunov numbers are defined as

$$\nu_X(m) = \inf\{a : \frac{\|B^t(m)\|}{a^t} \to 0 \text{ as } t \to \infty\} = \limsup_{t\to\infty} \|B^t(m)\|^{\frac{1}{t}}$$

and if $\nu_X(m) < 1$

$$\sigma_X(m) = \inf\{s : \|A^t(m)\| \, \|B^t(m)\|^s \to 0 \text{ as } t \to \infty\} = \limsup_{t\to\infty} \frac{\log \|A^t(m)\|}{-\log \|B^t(m)\|}$$

The rate of contraction in the direction normal to the manifold is measured by $\nu_X(m)$, while $\sigma_X(m)$ measures is the ratio of the exponential rate of contraction in the normal direction, and the exponential rate of contraction in M. Both $\nu_X(m)$ and $\sigma_X(m)$ are constant on orbits and depend only on the flow near the α set of m.

The following theorem will be used to provide conditions for persistent synchronization.

Theorem 2.5 (Fenichel, 1971) *Let X be a C^r vector field on $\mathbb{R}^n$ with an invariant, overflowing, compact, connected manifold with boundary $\bar{M} = M \cup \partial M$, properly embedded in $\mathbb{R}^n$. If $\nu_X(m) < 1$ and $\sigma_X(m) < 1/r$ for all $m \in M$ then for any C^r vector field Y in some C^1 neighborhood of X there is a manifold $\bar{M}_Y$ overflowing invariant under Y and C^r diffeomorphic to $\bar{M}$.*

There is a technical difficulty in applying this result to justify the definition 2.3 since the synchronization manifold is required to be stable and *inflowing* invariant. The definitions of the generalized Lyapunov numbers do not make sense in this case. The problem is resolved by the following construction:

Let X denote a vector field with flow F^t which leaves a manifold M invariant. Define $\phi : N \to \mathbb{R}^n$ by $\phi(m, v) = m + \exp_m v$. By a standard result of differential geometry [36] there exists a $\delta > 0$ such that ϕ maps $N^\delta = \{v \in N : |v| < \delta\}$ diffeomorphically onto a neighborhood of M.

Since the manifold M is inflowing invariant, we may define $M_\epsilon = F^\epsilon(M) \subset M$ so that M_ϵ is inflowing invariant. To each of the corresponding subsets of the normal bundle $N^{\delta,\epsilon} = \{(m, v) \in N_p^\delta : p \in M_\epsilon\}$ we can associate a neighborhood $T_\epsilon = \phi(N^{\delta,\epsilon})$.

Clearly $T_\epsilon \subset T_{\epsilon/2} \subset T_0$ (see figure 2) , and thus we can define a smooth function Φ on T_0 such that

$$\Phi(x) = \begin{cases} 0 & \text{if } x \in T_\epsilon, \\ 1 & \text{if } x \in T_0 - T_{\epsilon/2}. \end{cases}$$

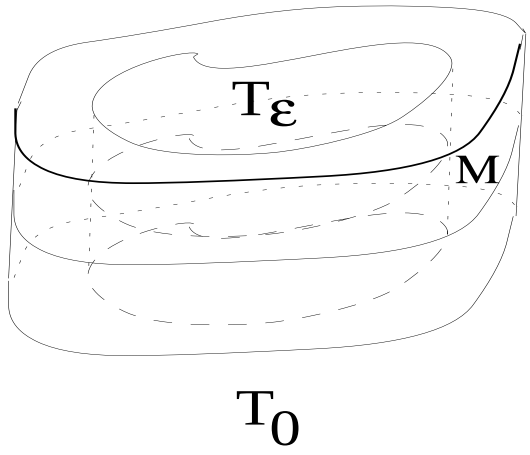

Figure 1: T_0 is a tubular neighborhood of the manifold M. T_0 contains T_ϵ, which is a tubular neighborhood of M_ϵ.

We will consider the vector field

$$\mathcal{X} = X + 2\Phi \cdot (kV - X) \tag{2}$$

where the vector field V is defined by $V_{\phi(m,v)} = -v$ and k is a constant which will be determined subsequently. Note that if $\tilde{X}$ is a small perturbation of X in T_0, then

$$\tilde{\mathcal{X}} = \tilde{X} + 2\Phi \cdot (kV - \tilde{X}) \tag{3}$$

is a small perturbation of $\mathcal{X}$. By construction, the manifold M is an overflowing invariant manifold under the flow induced by $\mathcal{X}$.

Let $\mathcal{T} = \{x \in T_0 : \Phi(x) > \frac{1}{4}\}$. Notice that the rate of contraction in the direction normal to M can be adjusted in $\mathcal{T}$ by changing the value of the constant k.

The vector field X points in on $\partial = \{x \in M : \phi(x) = \frac{1}{4}\}$. Also $\mathcal{X} = \frac{1}{2}X$ on ∂, so that by definition the vector filed $\mathcal{X}$ also points in on ∂. Therefore all orbits that leave M_ϵ in backwards time enter $\mathcal{T} \cap M$. Each $p \in \mathcal{T} \cap M$ has a coordinate neighborhood (x_i, y_i) inside $\mathcal{T} \cap M$ on which the vector field $\mathcal{X}$ can be expressed as

$$\begin{aligned} x_i' &= f_i(x_i, y_i) \\ y_i' &= g_i(x_i, y_i) - cy_i \end{aligned}$$

where $c > \frac{k}{4}$. The exponential contraction in the direction normal to the manifold may be bounded below by $K = \sup_{(x_i,y_i)\in\mathcal{T}} g_i(x_i, y_i) - \frac{1}{4}k$ Since k is an arbitrary constant, this implies that $\sigma(m)$, and $\nu(m)$ can be controlled for any m on orbits that leave the submanifold M_ϵ in backwards time.

If $\nu_X(m) < 1$ and $\sigma_X(m) < 1/r$ for all m on orbits that do not leave M_ϵ in backwards time, then the manifold M is r-hyperbolic and overflowing under the flow of $\mathcal{X}$ for sufficiently large values of k. Now if $\tilde{X}$ is a small perturbation of the X, then $\tilde{\mathcal{X}}$ as defined by equation (3) is a small perturbation of $\mathcal{X}$. Since M is r-hyperbolic, there is a manifold $\tilde{M}$ which is inflowing invariant under the flow of $\tilde{\mathcal{X}}$. Since $\tilde{M}$ is close to M and $\tilde{\mathcal{X}} = \tilde{X}$ in T_ϵ, the manifold $\tilde{M} \cap T_\epsilon$ is an inflowing invariant manifold under the flow of the vector field $\tilde{X}$. Thus we have proved the following

Proposition 2.6 *If $\nu_X(m) < 1$ and $\sigma_X(m) < 1/r$ for all m on orbits that do not leave M in backwards time then an inflowing submanifold of M containing an attractor of the flow F^t persists as an inflowing manifold under small perturbations.*

For an inflowing manifold it is more natural to calculate the generalized Lyapunov numbers in the direction of forward time. In particular $\nu_X(m)$ may be replaced by

$$\nu_X^*(m) = \limsup_{t\to\infty} \|\pi DF^t(m)|_N\|^{\frac{1}{t}}$$

measuring the contraction rate of the flow of a field X in forward time. The following proposition relating these two quantities is a direct consequence of Fenichel's Uniformity Lemma [37, Lemma II.2].

Proposition 2.7 *Let X and $\mathcal{X}$ be the vector fields constructed in the proof of proposition 2.6 and $\nu_{\mathcal{X}}(m) < 1$ and $\nu_X^*(m) < 1$ for all $m \in M$. Then for a sufficiently large k*

$$\sup_{m\in M} \nu_{\mathcal{X}}(m) = \sup_{m\in M} \nu_X^*(m)$$

3 The Relation Between Conditional Lyapunov Exponents and Generalized Lyapunov Numbers

The original papers of Pecora and Carroll [4] and Fujisaka and Yamada introduced conditional Lyapunov exponents, also known as sub-Lyapunov exponents, and normal or transversal Lyapunov exponents to measure the stability of the invariant manifold. The negativity of these numbers was used in many investigations as a criterion for the stability of the synchronization manifold. It was shown theoretically [45], and in numerical and physical experiments [46], [51] that a simple calculation of these quantities is insufficient for the prediction of synchronization.

To explain the relation between the transversal Lyapunov exponents as defined in [45] and the generalized Lyapunov numbers ν and σ we will assume that the coupled system has an attractor A contained in the synchronization manifold and that A is topologically transitive. The tangential and normal Lyapunov exponents can be defined as

$$\lambda_{\perp}(m,v) = \lim_{t\to\infty} \frac{1}{t} \log \|\pi_N DF^t(m)v\| \tag{4}$$

$$\lambda_{\|}(m,v) = \lim_{t\to\infty} \frac{1}{t} \log \|DF^t(m)w\| \tag{5}$$

when the limits exist and where $v \in N$ and $w \in TM$.

In [45] it is shown that both of these exponents exist μ-almost everywhere for any invariant measure supported on A. Moreover for each m, $\lambda_{\perp}(m,v)$ takes on one of the values $\lambda_{\perp}^1(m) \leq \ldots \leq \lambda_{\perp}^{s(m)}(m)$, and there is a related filtration of N_p equivalent to the case of ordinary Lyapunov exponents.

Let $SA = \{(p,v) : p \in A, v \in N \text{ and } \|v\| = 1\}$ denote the normal unit sphere bundle of the attractor A, and $\mathrm{Erg}_{F^t}(A)$ the set of all invariant probability measures supported on an attractor A.

Proposition 3.1 *Let M be an inflowing, invariant manifold for the flow of a vector field X satisfying the conditions of proposition 2.6 and containing a single topologically transitive attractor A of the flow. Moreover assume that $A = \cap_{t\in\mathbb{R}^+} F^t(M)$, that is M is a fundamental neighborhood of A. If $\sup_{\mu\in Erg_{F^t}(A)} \lambda_{\perp}^s(\mu) < 0$, or $\sup_{m\in M} \log \nu_X^*(m) < 0$ then*

$$\sup_{(m,v)\in SN} \limsup_{t\to\infty} \frac{1}{t} \log \|\pi DF^t(m)v\| = \sup_{\mu\in Erg_{F^t}(A)} \lambda_{\perp}^s(\mu) = \sup_{m\in M} \log \nu_X^*(m) \tag{6}$$

Equivalently, $\sup_{\mu\in Erg_{F^t}(A)} \lambda_{\perp}^s(\mu) \geq 0$ if, and only if $\sup_{m\in M} \log \nu_X^*(m) \geq 0$.

Proof: The first equality is contained in Theorem 2.8 of [46].

In the proof of proposition 2.6 it was shown that $\sup_{m\in M} \nu_{\mathcal{X}}(m)$ is achieved on orbits that do not leave M_ϵ. Since $A = \cap_{t\in\mathbb{R}^+} F^t(M_\epsilon)$ it follows that $\sup_{m\in M} \nu_{\mathcal{X}}(m)$ is achieved on A. By proposition 2.7 this implies that $\sup_{m\in M} \nu_X^* = \sup_{m\in A} \nu_{\mathcal{X}}$.

If $\nu_{\mathcal{X}} < a$ for all $m \in A$, then by Fenichel's Uniformity Lemma [37, Lemma II.2]

$$\|\pi DF^t(F^{-t}(m))\| < Ca^t$$

for all $m \in A$, and hence $\|\pi DF^t(m)\| < Ca^t$, so that $\nu_X^* < a$ for all $m \in A$. Similarly, if $\nu_X^* < a$ for all $m \in A$, then $\|\pi DF^t(m)\| < Ca^t$ for all $m \in A$. Therefore $\|\pi DF^t(F^{-t}(m))\| < Ca^t$ and $\nu_X^* < a$ for all $m \in A$. These inequalities imply $\sup_{m\in A} \nu_X^*(m) = \sup_{m\in A} \nu_{\mathcal{X}}(m)$.

It follows that

$$\sup_{m\in M} \nu_X^*(m) = \sup_{m\in A} \nu_{\mathcal{X}}(m) = \sup_{m\in A} \nu_{\mathcal{X}}(m) = \sup_{(m,v)\in SN} \limsup_{t\to\infty} \|\pi DF^t(m)v\|^{\frac{1}{t}}.$$

$\diamondsuit$

In other words the supremum of the transversal Lyapunov exponents on A is equal to the supremum of $\log \nu_X^*(m)$ on the manifold. Therefore the transversal Lyapunov exponents $\lambda_\perp(\mu)$ need to be negative for all invariant measures supported on the attractor A for the manifold to be stable. Even if the Lyapunov numbers of typical orbits in A are negative, it is possible that there are points $p \in A$ such that $\lambda_\perp(p) > 0$. In such cases the manifold M is not Lyapunov stable, and may not persist under perturbations. In such cases intermittent desychronization seems to be typical [50], [46].

Generally there are infinitely many ergodic invariant measures on the attractor A, which makes it difficult to directly calculate these quantities using numerical techniques. Fortunately the work of Hunt and Ott [47], [48], [49] suggests that the transversal Lyapunov exponents achieve their supremum on periodic orbits in A of low period. This gives some hope for computational estimates of the range of stability of synchronization manifolds.

4 Comments on Generalized Lyapunov Numbers

Since the synchronization manifold M is required to be smooth, the map ϕ providing the relation between the orbits of two synchronized systems is a diffeomorphism. If ϕ was only continuous, the Lyapunov dimensions of the attractors of the two systems could be different. Moreover, small errors in the measurements of one system could render ϕ useless in predicting the behavior of the second system making it physically irrelevant. There have been some misconceptions in the literature as to the correct quantities to measure the onset and stability of synchronization between chaotic systems. In this section we address these problems, and provide some cautionary examples.

Several methods for determining the asymptotic stability of the synchronization manifold have been based on the assumption that the asymptotic stability of the origin in the linearization of the equations describing the dynamics transversal to the manifold implies the transversal stability of the manifold itself. In general this is true only for coupled autonomous systems for which the network has bounded orbits. This result will follow as a corollary to a theorem about linear extensions.

Definition 4.1 *Let (X, p, B) be a vector bundle over a compact metric space B, and π^t and ρ^t be flows on X and B respectively. Then π^t is an* extension *of ρ^t if the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\pi^t"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ E \arrow[r, "\rho^t"] & E \end{tikzcd}$$

If $\pi^t : X_q \to X_{\rho^t(q)}$ is linear for all $t \in \mathbb{R}$, and $q \in B$, then π^t is a linear extension *of ρ^t. If $X = X_s \oplus X_u$ for some invariant vector subbundles X_s and X_u and*

$$\begin{aligned} \|\pi^t(x)\| &\leq d\|x\|e^{-\alpha t} \qquad x \in X_s, t > 0 \\ \|\pi^{-t}(x)\| &\leq d\|x\|e^{-\alpha t} \qquad x \in X_u, t > 0 \end{aligned} \tag{7}$$

then the extension is hyperbolic.

The following theorem is a weaker version of theorem III.3.4 in [34]

Theorem 4.2 *If $X = \{x \in X : \lim_{t\to\infty} \|\pi^t(x)\| = 0\}$, then the extension π^t of ρ^t is hyperbolic.*

Corollary 4.3 *Consider the coupled differential equations*

$$x' = f(x) \tag{8}$$

$$y' = A(x)y \tag{9}$$

where $A(x)$ is a matrix depending only on x. If $y = 0$ is asymptotically stable, and there exists a compact, inflowing manifold B for the flow of $x' = f(x)$, then (9) is uniformly asymptotically stable for all initial conditions $(x_0, y_0) \in B \times \mathbb{R}^n$

Proof: The flow on the entire space is a linear extension of the flow of the equation $x' = f(x)$, and hence we are in a position to use theorem 4.2. The only problem is that the set B is only positively invariant, while theorem 4.2 applies to fully invariant sets. This can be circumvented by creating a boundaryless double of the manifold B. Since B is compact and inflowing, it can be extended by a uniform amount in a backwards direction to $B_{-\epsilon} \supset B$. A copy of B_ϵ can be glued to B_ϵ along the boundary to create its boundaryless double D, which is a compact manifold without boundary [39]. We can leave the vector field unchanged in B, and use the same idea as in section 2 to extend it to a flow on D. We can now define a linear extension of the flow on D in an arbitrary way, as long as all points approach D in forward time, and it agrees with the original extension on $B \subset D$. The manifold D is compact and invariant, so that the extension is hyperbolic by theorem 4.2. It follows directly from the definition of hyperbolic linear extensions that the system (9) is uniformly asymptotically stable. $\diamondsuit$

Corollary 4.4 *Assume that there is a coordinate system in a neighborhood of a compact, inflowing manifold M in which the equations of motion take the form*

$$\begin{aligned} x' &= f(x) \\ y' &= g(x,y) \end{aligned} \tag{10}$$

where $g(x, 0) = 0$, so that M corresponds to the set $y = 0$. If the linearization of these equations around M

$$y' = D_y(x, 0)y \tag{11}$$

has $y = 0$ as an asymptotically stable point for any orbit in the manifold M, then the manifold M is uniformly asymptotically stable under the flow of (10).

Proof: This is a direct consequence of proposition 4.3, and the fact that the uniformly asymptotic stability of the origin in (11) implies the uniform asymptotic stability of the non-autonomous equation (10) as shown in theorem III.2.3 of [52]. $\diamondsuit$

In general it is not possible to conclude the stability of the synchronization manifold just by linearizing the equations around the synchronization manifold, as the following example shows.

Example 4.5 For the coupled pair of systems

$$x' = F(x,t)$$
$$y' = F(y,t) + E(x-y)$$

with $E(0) = 0$, the diagonal $\Delta = \{x, y \in \mathbb{R}^{2n} : x = y\}$ is an invariant manifold. If we let $w = x - y$, then the dynamics in the direction transversal to Δ are governed by

$$w' = F(x,t) - F(y,t) - E(x-y) \tag{12}$$

This equation can be linearized around Δ to

$$w' = [D_x F(x,t) + DE(0)]w \tag{13}$$

There are instances in the literature where it is assumed that the asymptotic stability of the origin in (13) implies the asymptotic stability of the origin of (12). Without the additional assumption of *uniform asymptotic stability* this is in general not true.

The following counterexample is an adaptation of a famous counterexample by Peron [52]. Consider the following pair of coupled systems:

$$\begin{aligned} x_1' &= -ax_1 \\ x_2' &= (-2a + \sin(\ln t) + \cos(\ln t))x_2 \\ y_1' &= -ay_1 \\ y_2' &= (-2a + \sin(\ln t) + \cos(\ln t))y_2 - (x_1 - y_1)^2 \end{aligned}$$

with $a > \frac{1}{2}$. The independent solutions to the linearization of these equations around the diagonal are

$$\begin{aligned} w_1(t) &= \exp(-2at) \\ w_2(t) &= \exp(-2at + t\sin(\ln t)). \end{aligned} \tag{14}$$

Therefore the linearized system (13) has the origin as an asymptotically stable point. The solutions of the equations describing the driving system (x_1, x_2) have the same solutions, so that we can choose an $\epsilon > 0$ and $\Delta_\epsilon = \{(x, y) \in \Delta : |x| < \epsilon\}$ as a candidate for the synchronization manifold.

Notice that the origin is not uniformly asymptotically stable for equations (14). The full equations describing the motion transversal to the synchronization manifold are

$$\begin{aligned} w_1' &= -aw_1 \\ w_2' &= (-2a + \sin(\ln t) + \cos(\ln t))w_2 + w_1^2. \end{aligned} \tag{15}$$

It can be shown that these equations have the origin as an unstable fixed point. It follows that the response system y will not synchronize to the drive x despite the asymptotic stability of the linearized system.

By the Fenichel's Uniformity Lemma if $a > \sup_{m \in M} \log \nu^*_X(m)$, then there are constants $\hat{a} < a$ and c such that $\|\pi DF^t(m)|_N\| < c\hat{a}^t$ for all $m \in M_\epsilon$, and $t \geq 0$. Therefore despite the fact that ν^* depends on the asymptotics of the dynamics, it also provides an *uniform* estimate on the rate of contraction normal to the synchronization manifold.

The normal Lyapunov exponents and $\nu^*(m)$ only determine the stability of the manifold in a possibly small neighborhood. In pathological situations it is possible that the basin of attraction is very small in the normal direction, so that a small amount of noise may desynchronize the systems.

Blakely, et. al. [53] also provide an example of two phase locked periodically oscillating systems coupled in such a way that the stable eigenvectors of the fixed point of the Poincaré section are non-normal. In this situation the ϵ-spectrum of the return matrix is large, and hence small perturbations of the system have a large influence on its stability. Experimental observation indicate that two physical systems of this type are very difficult to synchronize, even when the transversal Lyapunov exponents are found computationally to be negative. It is unclear in which instances desynchronization is due to non-normality, to the existence of positive normal Lyapunov exponents, or to both.

Remark 4.6 Due to the finite precision of numerical calculations, two identical coupled systems starting at different points in phase space may become identically synchronized in finite time. This follows from the fact that the exponentially decaying distance between them may eventually be smaller than the numerical precision at which the calculations are carried out, and hence approximated by 0. This is referred to as *hard locking*, and is a purely numerical artifact. It is possible that a true orbit intermittently leaves the synchronization manifold, which will not be observed if hard-locking occurs. In numerical simulations in which intermittent desynchronization is expected to occur, we can either add a small amount of noise to the system, or introduce a small difference between the two systems to prevent numerical hard locking. Hard locking will in general not occur in the case of generalized synchronization.

The importance of the condition $\sigma_X < 1$ in theorem 2.5 was demonstrated by Hunt, Ott and Yorke [54] who have shown that in the case of unidirectionally coupled system if $\sigma(p) > 1$ for any point on the synchronization manifold M, the manifold will generally persist only as a non-differentiable, Hölder continuous manifold under perturbations. Moreover, the Hölder exponent of the synchronization manifold also depends on the strength of the coupling between the systems. These results will be discussed further in section 5. A similar example of an attracting torus which is not normally hyperbolic and bifurcates into an attractor which is not even a topological manifold is provided in [55].

5 Extensions

The definition of chaotic synchronization can be directly extended to cover more complex situations in larger networks of systems. In keeping with the literature on networks of coupled oscillators, we will refer to each unit in a collection as a system, while the entire collection will be called a network. Thus two coupled oscillators form a network of two systems.

Consider a network of coupled systems $x_1, x_2, \ldots, x_N$ where $x_i \in \mathbb{R}^{n_i}$. The entire system evolves in $\mathbb{R}^K$, where $K = \sum_{i=1}^N n_i$. We can consider the natural projections $\Pi_i : \mathbb{R}^{Nn} \to \mathbb{R}^{n_i}$ onto the phase space of the networks i-th system. Assume that the evolution of the system is determined by

$$x_i' = f_i(x_1, \ldots x_N) \qquad f_i : \mathbb{R}^K \to \mathbb{R}^{n_i}. \tag{16}$$

If only a subset of the systems comprising the network are synchronized, partial synchronization. Following definition 2.1 partial synchronization is defined in terms of the existence of a diffeomorphism from the neighborhood of the attractor of one system in the network to the attractor of another. Such a diffeomorphism should be independent of the behavior of the remainder of the network. and for this purpose we introduce the following "section" of $\mathbb{R}^K$

$$S_{i,j}(c) = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^K : (x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_N) = c\} \qquad c \in \mathbb{R}^{K-n_i+n_j}$$

where $\hat{x}_i$ means that the i-th vector is omitted.

Definition 5.1 *The system i is synchronized to the system j in the network (16) if there exists a compact, smooth synchronization manifold M with boundary which is invariant under the flow, inflowing, locally attracting, and such that the projection Π_j of $M \cap S_{i,j}(c)$ is smooth and one-to-one for all $c \in \mathbb{R}^{K-n_i+n_j}$. The synchronization is stable if M is r-normally hyperbolic for $k \geq 1$. A network for which these conditions hold for a subset of L systems where $L < N$ is called* partially synchronized*.*

Example 5.2 Consider two diffusively coupled Lorenz systems with a coupling sufficiently strong to put them in a synchronization regime [41]. This small network can now be expanded by adding more systems. If the connections are sufficiently weak, the small interactions will not destroy the synchronization manifold, oscillators 1 and 2 will remain synchronized, and thus the network will be partially synchronized.

Definition 5.1 is rather weak and covers many types of coherent states in networks, as well as master-slave synchronization introduced in [4]. The following example shows that this relation is not necessarily reflexive.

Example 5.3 Consider the system

$$\begin{aligned} x' &= g(x), \qquad x \in \mathbb{R}^n, \qquad n > 1 \\ y' &= -ky + \epsilon\, c(x), \qquad y \in \mathbb{R} \end{aligned} \tag{17}$$

where g and c are C^1 and the unit ball $B_1 \subset \mathbb{R}^n$ is inflowing invariant for equation (17). For small ϵ and sufficiently large k the manifold $M = B_1 \times \{0\}$ persists as an invariant manifold of the coupled system, and moreover its projection onto the phase space of x is smooth and one-to-one. However the projection onto the one-dimensional phase space of y will in general not be even finite-to-one. Therefore for small ϵ, y will be synchronized to x, but x will not be synchronized to y.

As this example indicates, we cannot expect mutual synchronization given two coupled systems which require a different number of variables to be fully described.

The most direct way of showing existence of invariant manifolds is applicable to networks with certain symmetries. Let Γ be a group acting on the phase space of the entire network $\mathbb{R}^K$ so that the induced action on the tangent space leaves the vector field defined by (16) invariant. If $\mathrm{Fix}(\Gamma) = \{x \in \mathbb{R}^K : \Gamma x = x\}$ is a manifold then it will be invariant under the flow of the equations (16). For $\mathrm{Fix}(\Gamma)$ to be a synchronization manifold it needs to satisfy the remaining conditions given in 5.1.

Notice that if Γ has a subgroup S, then $\mathrm{Fix}(S) \supset \mathrm{Fix}(\Gamma)$. It can happen that as parameters are changed, the manifold $\mathrm{Fix}(\Gamma)$ looses stability, while $\mathrm{Fix}(S)$ remains stable. This occurs when $\mathrm{Fix}(\Gamma)$ looses stability in $\mathrm{Fix}(S)$, but not in the direction normal to $\mathrm{Fix}(S)$. In this case synchronization in large clusters of oscillators breaks down to synchronization of smaller clusters. In particular, if Γ has many subgroups $S_1, \ldots S_n$, the subgroups of smaller order can correspond to invariant manifolds of higher dimensions, and therefore to smaller synchronization clusters. This situation was examined recently by Ashwin and Terry [44].

The following is a physically relevant example of strict partial synchronization:

Example 5.4 In [43] a system of three lasers modeled by equations of the following was analyzed

$$\begin{aligned} x_1' &= f(x_1, x_2) \\ x_2' &= g(x_1, x_2, x_3) \qquad\qquad x_1, x_2, x_3 \in \mathbb{R}^n \\ x_3' &= f(x_3, x_2) \end{aligned} \tag{18}$$

where $g(x_1, x_2, x_3) = g(x_3, x_2, x_1)$. The system is invariant under the transformation $\mu(x_1, x_2, x_3) = (x_3, x_2, x_1)$ so that the manifold $N = \mathrm{Fix}(\mu) = \{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : x_1 = x_3\}$ is invariant. If N contains an inflowing attracting submanifold M, the conditions of definition 5.1 are satisfied. It is clear that in this case M is the graph of the identity map between the phase spaces of system x_1 and x_3.

On M the equations reduce to

$$x_1' = f(x_1, x_2) \quad x_2' = g(x_1, x_2, x_1) \tag{19}$$

and without further assumptions this system does not have any further invariant manifolds. In the particular case considered in [43] it is shown that in particular $\{(x_1, x_2, x_3) \in \mathbb{R}^{3n} : x_1 = x_2 = x_3\}$ is not an invariant manifold, and hence synchronization of all three lasers is not possible.

Numerically it is observed that the two outer lasers x_1 and x_3, are synchronized, while they do not synchronize with x_2. From a physical viewpoint this might seem peculiar since x_1 and x_3 are not coupled to each other directly, but only indirectly through the middle laser x_2. A schematic representation of this situation is given in figure 5.4

The condition that the projections Π_i are one-to-one can be further weakened.

Definition 5.5 *If the projection Π_j of $M \cap S_{i,j}(c)$ in definition 5.1 is smooth and n-to-one for all $c \in \mathbb{R}^{K-n_i+n_j}$ then system i is* partially phase locked *to system j. If, moreover, the projections Π_i of $M \cap S_{i,j}(c)$ is smooth and m-to-one for all $c \in \mathbb{R}^{K-n_i+n_j}$ then the systems i and j are $n : m$* phase locked.

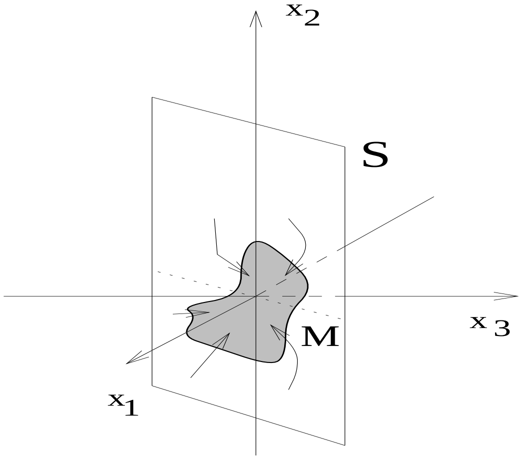

Figure 2: A schematic representation of the situation in example 5.4. The shaded area corresponds to the synchronization manifold.

In this case the manifold M is only locally a graph of a smooth map from system i to j. A simple example is the $n : m$ phase locking of phase oscillators. In [22], the auxiliary method was used to show that a chaotically driven chaotic circuit synchronizes to the driving signal. In one case, the projection of the synchronization manifold onto the phase space of the driven system is 2:1. In more general instances of phase locking of chaotic systems such as the ones examined in [23] and [24], a synchronization manifold may not exist, and such examples may not be covered by the present definitions. The case of $n : m$ synchronization between unidirectionally coupled systems is discussed further in section 6.

Remark 5.6 Definitions 5.1 and 5.5 apply only to autonomous systems. However a network of periodically or quasi-periodically forced systems evolves $\mathbb{T}^N \times \mathbb{R}^K$ for some integer N. Such a network will be (partially) synchronized if there exists a submanifold $M = \mathbb{T}^N \times M_0$ where M_0 satisfies the conditions of definition 5.1 or 5.5. The synchronization will again be stable if M is r-normally hyperbolic.

The case of more general non-autonomous systems is more problematic since a synchronization manifold may no longer be compact, and thus the results from [37] do not carry over directly. Extensions of the theory to non-compact invariant manifolds do exist, but some further uniformity in the vector field and the invariant manifold need to be assumed [56].

6 Unidirectionally Coupled Systems as Extensions

Two systems are coupled unidirectionally if they are described by equations of the form

$$x' = f(x) \qquad y' = g(x, y) \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m \tag{20}$$

It was proved in [41] and [42] that, under appropriate conditions, there exists a smooth, local change of coordinates in some neighborhood Ω of M which takes a synchronized bidirectionally coupled system into a unidirectionally coupled system. In the next two sections we will examine unidirectional couplings with the understanding that under appropriate conditions these results can be carried over to more general situations.

In the case of unidirectional coupling the existence of a synchronization manifold can be proved as a direct consequence of results on extensions of flows which can be found in [57] and [34]. Denote by Φ^t the flow of the entire system (20) on $\mathbb{R}^{n+m}$, and by Ψ^t the flow of the driving system x on $\mathbb{R}^n$. If $\mathbb{R}^{n+m}$ is viewed as a trivial vector bundle over $\mathbb{R}^n$, then it is immediate that Φ^t is an extension of Ψ^t in the sense of definition 4.1.

Using this notation and Theorem 3.1 in [57] we immediately have the following result giving sufficient conditions for the existence of a *possibly non-differentiable* synchronization manifold.

Theorem 6.1 *If there exists an invariant, inflowing, compact manifold Ω for the flow Ψ^t in $\mathbb{R}^n$, a $T_0 > 0$ and $K < 1$ such that for every $x \in \Omega$ and y_1, y_2*

$$|\Phi^{T_0}(x, y_1) - \Phi^{T_0}(x, y_2)| < K|y_1 - y_2| \tag{21}$$

then there exists a unique continuous invariant section $\sigma : \Omega \to \Omega \times \mathbb{R}^m$

In other words, if the conditions of the theorem are satisfied there exists a continuous function Σ with graph σ taking orbits of system x to orbits of system y.

Let $I_c = \{x \in \mathbb{R}^m : |x| < c\}$ and assume that the box $\Omega \times I_c$ is inflowing under the flow of the full system (20). If $\lim_{t\to\infty} |\Phi^t(x, y_1) - \Phi^t(x, y_2)| = 0$ for all $x \in \Omega$ and $|y_1|, |y_2| < c$ then we call the system *asymptotically stable* in $\Omega \times I_c$ with respect to the projection p.

Corollary 6.2 *Assume that the system (20) has an attractor A, is asymptotically stable in $\Omega \times I_c$ with respect to p, and there exists a T_0 such that condition (21) is satisfied on set $\Omega \times I_c$. Then there exists a continuous function $F : \mathbb{R}^n \to \mathbb{R}^m$ taking the orbits of system x in the neighborhood of $p(A)$ to the orbits of system y.*

Remark 6.3 There is some confusion in the literature about the properties of the function F. There were claims that, if the system is asymptotically stable on $\Omega \times I_c$ without necessarily satisfying condition (21), then an invariant, possibly discontinuous section of the normal bundle exists. In general this may not be true since the function F may be multivalued [60]. Asymptotic stability is in general weaker than condition (21). If the system is assumed to be only asymptotically stable in a compact, positively invariant set $B \subset E$, the situation is more complicated, and will be addressed in the next section.

Theorem 6.1 does not imply that the map F is smooth, and hence its graph will not necessarily be a synchronization manifold as defined in section 2. The next result shows that the larger the contraction within the fibers, the higher the regularity of the synchronization manifold $M = \text{graph}(F)$.

Definition 6.4 *Let Ω be an open subset of $\mathbb{R}^n$ with compact closure, overflowing invariant under Ψ^t. The flow Ψ^t is* k-contracting *over Ω if there exist numbers $c > 0$ and $0 < \mu < 1$ such that for $s = 0, 1, \ldots, k$ the following relation holds*

$$\sup_{x\in\Omega}(\|D\Psi^{-t}(x)\|^s \|D\Phi^t|_{T_{\Psi^{-t}(x)}}\|) \leq c\mu^t \quad \textit{for all} \quad t > 0 \tag{22}$$

where

$$\|D\Phi^t|_{T_{\Psi^{-t}(x)}}\| = \sup\{\|D\Phi^t(w)\| : w \in T_\nu p^{-1}(x) \text{ where } \nu \in p^{-1}(x), \text{ and } \|w\| < 1\}. \tag{23}$$

This definition simply says that the flow Ψ^t is k-contracting if the rate of contraction within the fibers is k times *exponentially* stronger than the rate of contraction of the flow in the base Ω.

Theorem 6.5 (Hirsch, Pugh, Shub 1977) *If the flow Ψ^t is k-contracting over Ω, then there exists a unique C^k smooth Ψ^t invariant section $F : \mathbb{R}^n \to \mathbb{R}^{n+m}$. In other words system y is synchronized to system x, and the synchronization manifold is graph(F).*

Remark 6.6 Here we again encounter the same technical difficulty that we encountered in section 2, since the manifold Ω is assumed to be overflowing. To apply these results to the case where Ω contains an attractor of the first system we can apply the collaring construction used in section 2. This shows that if Ω is inflowing, condition (22) needs to be satisfied only on orbits that do not leave Ω in backwards time for theorem 6.5 to imply the existence of a unique synchronization manifold $M = \text{graph}(F)$.

If the map F is not differentiable, we can use its Hölder exponent as a measure of regularity. Roughly speaking, the higher the Hölder exponent of the map, the more F looks like a smooth map. Results about the Hölder regularity of a potentially non-differentiable synchronization manifold are discussed in [54].

Example 6.7 A direct method of determining generalized synchronization in physical systems, and numerical models is the auxiliary method introduced in [7]. The method is essentially an application of the theory outlined above.

Given a unidirectionally coupled system $x' = f(x)$, $y' = f(x, y)$, the idea is to consider the auxiliary system $z' = f(x, z)$. It is difficult to experimentally determine whether the systems x, and y are synchronized in a generalized sense. However if the systems y and z synchronize identically, we may conclude that $\lim_{t\to\infty} |\phi^t(x, y_0) - \phi^t(x, z_0)| = 0$, since the flow of the systems (x, y) and (x, z) is the same. If the system x has an attractor, the orbits of the extended system remain uniformly bounded, then there exists a map F taking the attractor of system x to an attractor of system y. If the contraction rate is uniform in the sense of equation (21), then the conditions of corollary 6.2 are satisfied, and F is continuous.

The regularity of the map F can then be checked using theorem 6.5 and results in [54]. In particular the exponential decay of $|\phi^t(x, y_0) - \phi^t(x, z_0)|$ can be compared to the contraction rates within the attractor $p(A)$ of the driving system. Non-typical orbits of the attractor of system x may play an important role in such comparisons. In particular the contraction rates would have to be compared for all orbits in the attractor to obtain a true bound on the regularity. This will typically be impossible to do numerically, and analytic techniques, or results similar to those announced in [48] and [49] need to be used.

Also note that for diffusive couplings, an increase in coupling strength is going to result in a higher regularity of the synchronization manifold. For instance consider the following equations

$$x' = f(x) \quad y' = g(y) + K(x - y) \quad z' = g(z) + K(x - z) \tag{24}$$

where z plays the role of an auxiliary system, and the driving system has an attractor A. Let $\eta = x - y$ so that

$$\eta' = f(x) - g(y) - K\eta. \tag{25}$$

Let $\bar{\Omega}$ be a compact set containing A, and choose $c > 0$ so that $c > \sup\{|x| : x \in \bar{\Omega}\}$.

We want the set $\bar{\Omega} \times I_c$ to be inflowing under the flow of this dynamical system. By compactness K can be defined so that

$$K > \sup_{|y|=c, x\in\Omega} \left[\frac{|f(x) - g(y)|}{|x - y|}\right] \tag{26}$$

then η in equation (25) will be decreasing for all $|y| = c$ and $x \in \Omega$, which means that $\bar{\Omega} \times I_c$ is inflowing.

Let $\chi = y - z$, so that $\chi' = g(y) - g(z) - k\chi$. Expanding in a Taylor series around y gives

$$\chi' = [D_y g(y) - K]\chi + f(y, z) \tag{27}$$

Since $f(y, z) \to 0$ uniformly on the compact set $\bar{\Omega} \times I_c$, by theorem 2.4 of [52] it is sufficient to examine the linear part of equation (27). An application of Duhamel's Formula and Gronwall's Inequality shows that when $K > \gamma = \sup_{y\in I_c} |D_y g(y)|$ the contraction in the fibers can be made arbitrarily strong by choosing a sufficiently large K. Moreover the rate of contraction is independent of the base point of the fiber and can be controlled uniformly throughout $\bar{\Omega} \times I_c$. Therefore system y can be synchronized to system x with a synchronization manifold of arbitrarily large smoothness by choosing a sufficiently large K.

7 $n:1$ Synchronization in Unidirectionally Coupled Systems

Next we consider a unidirectionally coupled system for which the response is uniformly asymptotically stable in a set $B \subset E$. Under these assumptions, there may not exist a function F taking orbits from the drive to the response system, The existence of a more general correspondence will be proved under some mild restrictions. The results of this section were pointed out by I. Bronshteyn [59].

Definition 7.1 *Let (E, p, X) be an vector bundle over X, and the flow Φ^t on E an extension of the flow Ψ^t on X. Let A be an attractor of the flow Ψ^t, $\bar{\Omega}$ a compact inflowing set containing A, and B a forward invariant subset of E such that $p(B) = \Omega$. Then B is* uniformly asymptotically stable *with respect to p if:*

1. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $a_1, a_2 \in B$ with $p(a_1) = p(a_2)$ and $d(a_1, a_2) < \delta$ we have $d(\Phi^t(a_1), \Phi^t(a_2)) < \epsilon$ for all $t > 0$.*

2. *For every $\epsilon > 0$, there exists a $\gamma > 0$ such that if $a_1, a_2 \in B$, $p(a_1) = p(a_2)$ and $d(a_1, a_2) < \gamma$, then $d(\Phi^t(a_1), \Phi^t(a_2)) < \epsilon$ for all t.*

These two conditions imply that $\lim_{t\to\infty} d(\Phi^t(a_1), \Phi^t(a_2)) = 0$.

In what follows it will be assumed that A is connected and contains negative semi-orbits that are dense in A. Let M be a minimal subset of A, so that for any $x \in \Omega$ the orbit $\{\Psi^t(x) : t \in \mathbb{R}\}$ is dense in M. Let $\tilde{A}$ be a connected attractor contained in B, and

$$\tilde{A}_0(M) = \tilde{A} \cap p^{-1}(M). \tag{28}$$

The following nontrivial lemma is proved in [58].

Lemma 7.2 *Under the assumptions of the preceding paragraphs, the map* $p : \tilde{A}_0(M) \to M$ *is a finite-to-one, local homeomorphism.*

In general, the map $p : \tilde{A}_0(M) \to M$ is not one-to-one. Consider, for instance, the case of a periodically driven phase oscillator which is 2:1 phase locked to the driving signal. If B is any closed set contained in the basin of the attracting orbit then the conditions of lemma 7.2 are satisfied. In this case the projection $p : \tilde{A}_0(M) \to M$ is two-to-one.

Lemma 7.2 can be used to prove some properties of the projection from the attractor of the coupled system $\tilde{A}$ to the attractor of the driving system A.

Theorem 7.3 *For each point* $a \in A$ *the set* $p^{-1}(a) \cap \tilde{A}$ *consists of a finite number of points. Moreover, the function* $a \to card[p^{-1}(a) \cap \tilde{A}]$ *is constant on a residual subset of* A.

Proof: Note that for each two points $a_1, a_2 \in B$ such that $a_1 \neq a_2$, and $p(a_1) = p(a_2)$ there exists a $\delta(a_1, a_2)$ such that

$$\inf_{t<0} d(\Phi^t(a_1), \Phi^t(a_2)) \geq \delta(a_1, a_2). \tag{29}$$

We can prove this as follows: Let $\epsilon = d(a_1, a_2) > 0$, and δ be as in definition 7.1. Seeking a contradiction, assume that there exists a $t_0 < 0$ such that $d(\Phi^{t_0}(a_1), \Phi^{t_0}(a_2)) < \delta$. If we choose $b_1 = \Phi^{t_0}(a_1)$ and $b_2 = \Phi^{t_0}(a_2))$, then $p(b_1) = p(b_2)$ and $d(b_1, b_2) < \delta$. Therefore $d(\Phi^t(b_1), \Phi^t(b_2)) < \epsilon = d(a_1, a_2)$ for all $t > 0$. Choosing $t = -t_0$ gives a contradiction.

Let $x_1, x_2 \in A$ be two points whose negative semiorbits $\{\Psi^t(x_1) : t \leq 0\}$ and $\{\Psi^t(x_2) : t \leq 0\}$ are dense in A. There exists a sequence $\{-t_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \Psi^{-t_n}(x_1) = x_2$. By compactness, there is a subsequence of the sequence of maps $\{\Phi^{-t_n} : B \to B\}_{n=0}^{\infty}$ which converges to a map $\chi : B \to B$. The map χ is not necessarily continuous. However since any two points $a_1, a_2 \in p^{-1}(x_1)$ remain a distance $\delta(a_1, a_2)$ apart along the backwards orbit, the restriction

$$\chi|_{p^{-1}(x_1)\cap\tilde{A}} : p^{-1}(x_1) \cap \tilde{A} \to p^{-1}(x_2) \cap \tilde{A} \tag{30}$$

is injective. Since the same argument can be carried out with x_2 in place of x_1 it follows that $p^{-1}(x_1) \cap \tilde{A}$ and $p^{-1}(x_2) \cap \tilde{A}$ have the same cardinality. Since it was assumed that A contains negative semi-orbits that are dense in A, the set of all orbits with negative orbits that are dense in A is residual in A.

The closure of the negative semiorbit of an arbitrary point $x \in A$ contains a minimal set M. By an argument identical to the one given in the preceding paragraph card$[p^{-1}(x) \cap \tilde{A}] <$ card$[p^{-1}(x_0) \cap \tilde{A}]$. However by lemma 7.2 card$[p^{-1}(x_0) \cap \tilde{A}]$ is finite. $\diamondsuit$

Remark 7.4 In a general situation we therefore expect that the response is $n : 1$ phase locked to the driving signal. An example of a response that is 2:1 phase locked to a chaotic driving signal is given in [22]. In general, theorem 7.3 might be the best possible. Examples in [60] show that even when the response is asymptotically stable, $\mathrm{card}[p^{-1}(x) \cap \tilde{A}]$ may not be independent of x. Thus it is possible that the response system is $n(x) : 1$ phase locked to the drive, where $n(x)$ depends on $x \in A$.

Remark 7.5 Assume that the attractor A contains periodic orbits, $B = \bar{\Omega} \times I_c$, and B is uniformly asymptotically stable. Then for any $x \in A$ which lies on a periodic orbit $\mathrm{card}[p^{-1}(x) \cap \tilde{A}] = 1$. This follows from the fact that I_c is connected, and thus definition 7.1 implies that for all $a_1, a_2 \in B$ with $p(a_1) = p(a_2)$ we have $\lim_{t\to\infty} d(\Phi^t(a_1), \Phi^t(a_2)) = 0$. However if $\mathrm{card}[p^{-1}(x) \cap \tilde{A}] > 1$, and T_0 is the period of the orbit $\mathcal{O}(x)$, then $d(\Phi^{nT_0}(a_1), \Phi^{nT_0}(a_2)) = d(a_1, a_2) > 0$ for all $n \in \mathbb{Z}$.

By an argument identical to the one given in the proof of theorem 7.3, only points with no periodic orbit in the closure of their backwards orbits could have $\mathrm{card}[p^{-1}(x) \cap \tilde{A}] > 1$. This fact seems to have been misinterpreted in some references.

8 Synchronization from the Viewpoint of Geometric Singular Perturbation Theory

So far we have seen that the existence of a synchronization manifold can be proved using the symmetries of the system and by treating unidirectinally coupled systems as extensions. In this section we show that geometric singular perturbation theory [64] can be used to prove synchronization between diffusively coupled systems for strong coupling.

This approach follows from the observation that in the limit of infinite coupling strength a diffusively coupled network frequently possesses a normally hyperbolic synchronization manifold. Perturbing from the limit of infinite coupling strength we can conclude that the network is stably synchronized for large couplings. Moreover this approach also allows us to analyze the equations of motion on the synchronization manifold as a perturbation of the equations obtained in the limit of infinite coupling strength.

Consider again the network (16). Following definition 5.1 we will say that a compact manifold M with boundary is *diagonal-like* if the projection Π_j of $M \cap S_{i,j}(c)$ is smooth and one-to-one for all $c \in \mathbb{R}^{K-n_i+n_j}$ for at least one pair (i, j).

Proposition 8.1 *Assume M_0 is diagonal-like, and that there is a change of coordinates such that in a compact neighborhood of M_0, equation (16) can be written as*

$$\begin{aligned} x' &= f(x, y) - \Omega(x, y, c) \\ y' &= g(x, y) \end{aligned} \tag{31}$$

where c is the coupling parameter and $M_0 = \{(x, y) \in \mathbb{R}^K : x = 0\}$. Also assume that there exists a C^1 monotonically increasing function $m(c)$, such that $\lim_{c\to\infty} m(c) = \infty$ and $\lim_{c\to\infty} \frac{\Omega(x,y,c)}{m(c)} = \omega(x, y)$ uniformly in the C^1 topology, and $\omega(0, y) = 0$. If $\omega_i(x, y) > 0$ for all components ω_i of ω and all $(x, y) \in K$, then for sufficiently large coupling c there exists a manifold M_ϵ locally invariant under the flow of (31), and $\mathcal{O}(\epsilon)$ close to M_0.

Proof: Define $\epsilon = 1/m(c)$. After dividing both sides of equation (31) by $m(c)$ we have

$$\begin{aligned} \epsilon x' &= \epsilon f(x,y) - \frac{\Omega(x,y,c)}{m(c)} \\ y' &= g(x,y). \end{aligned} \tag{32}$$

which can be viewed as a perturbation problem from $\epsilon = 1/m(c) = 0$. In this limit the fast equations have the form

$$\dot{x} = -\omega(x,y) \qquad\qquad \dot{y} = 0$$

and hence the manifold $M_0 = \{(x,y) : x = 0\}$ is normally hyperbolic for the fast equations if $\omega_i(x,y) > 0$ for all components ω_i of ω and all $(x,y) \in K$. By fundamental results of geometric singular perturbation theory [64], this implies that there exists a locally invariant manifold M_ϵ which is $\mathcal{O}(\epsilon)$ close to M_0. $\diamondsuit$

To show that M_ϵ is a synchronization manifold it needs to be shown that it is inflowing. Since $M_0 = \{(x,y) \in \mathbb{R}^K : x = 0\}$, it follows that $M_\epsilon = \{(x,y) \in \mathbb{R}^K : x = h^\epsilon(y)\}$ where $h^\epsilon(y) = \mathcal{O}(\epsilon)$. The slow equations corresponding to system (31) when $\epsilon = 0$ have the form

$$x' = 0 \qquad\qquad y' = g(0,y). \tag{33}$$

It follows that for ϵ sufficiently small the equations (31) have the form $y' = g(h^\epsilon(y), y)$ on M_ϵ, and can be studied using ordinary perturbation techniques. In particular if the conditions of proposition 8.1 are satisfied and M_0 is an inflowing manifold for (33), then M_ϵ is a synchronization manifold.

This approach is very similar to the one adopted in [9] for systems with linear couplings. The main advantage of using geometric singular perturbation theory is that it is not necessary to assume that M_ϵ is an inflowing manifold, rather this can be concluded from the fact that M_0 is inflowing.

The following examples demonstrates these ideas

Example 8.2 Consider a pair of coupled systems

$$\begin{aligned} x' &= f(x,y) - c(x-y) \\ y' &= g(x,y) - c(y-x) \qquad\qquad x, y \in \mathbb{R}^n \end{aligned} \tag{34}$$

Without any assumptions on f and g we can not use symmetry to prove the existence of an invariant manifold. However if we let $\eta = x - y$, and $\chi = x + y$, then the equations (34) can be written as

$$\begin{aligned} \epsilon \eta' &= \epsilon(\tilde{f}(\chi,\eta) - \tilde{g}(\chi,\eta)) - 2\eta \\ \chi' &= \tilde{f}(\chi,\eta) + \tilde{g}(\chi,\eta) \end{aligned} \tag{35}$$

where $\epsilon = 1/c$, and $\tilde{f}(\chi,\eta) = f(\frac{\chi+\eta}{2}, \frac{\chi-\eta}{2})$, and $\tilde{g}(\chi,\eta) = g(\frac{\chi+\eta}{2}, \frac{\chi-\eta}{2})$. Taking the limit $c \to \infty$, we get the pair of equations

$$\begin{aligned} \eta &= 0 \\ \chi' &= \tilde{f}(\chi,\eta) + \tilde{g}(\chi,\eta). \end{aligned} \tag{36}$$

In particular, if there is a compact inflowing submanifold for the flow of

$$\chi' = \tilde{f}(\chi, 0) + \tilde{g}(\chi, 0) \tag{37}$$

then it will persist as such under small perturbations. Since the invariant manifold described by $\eta = 0$ corresponds to $M_0 = \{(x, y) : x = y\}$, we can conclude that for large coupling strengths the two systems will be nearly identically synchronous.

There is nothing special about linear couplings. Assume that a network of systems can be described by the equation

$$z' = f(z) + K(z, c) \tag{38}$$

where $K : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is some coupling function. Let l be the codimension of the manifold M_0 in the phase space of the network. Assume that M_0 is a manifold invariant under the flow of $x' = K(x, c)$ for all values of c, and the neighborhood of M_0 can be smoothly foliated by l-dimensional leaves which are invariant under the flow of $x' = K(x, c)$. If this foliation provides a coordinate system in a neighborhood of M_0 then there is a natural way to write the system in the form of equation (31) in this neighborhood. In this case the function Ω will depend only on x and c. If $\Omega(x, c)$ satisfies the conditions of proposition 8.1, then we can conclude that synchronization occurs.

Example 8.3 Consider the following Rössler system driven by a Lorenz system

$$\begin{aligned} x_1' &= \sigma(y_1 - x_1) & x_2' &= -y_2 - z_2 - c(x_2 - (x_1^2 + y_1^2)) \\ y_1' &= \rho x_1 - y_1 - x_1 z_1 & y_2' &= x_2 + a y_2 - c(y_2 - (y_1^2 + z_1^2)) \\ z_1' &= x_1 y_1 - \beta z_1 & z_2' &= b + z_2 x_2 - d z_2 - c(z_2 - (x_1^2 + z_1^2)) \end{aligned}$$

where all constants are chosen so that both systems are in the chaotic regime. In this case the manifold M_0 is a compact submanifold of the set

$$S = \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : x_2 = x_1^2 + y_1^2, y_2 = y_1^2 + z_1^2, z_2 = x_1^2 + z_1^2\}. \tag{39}$$

For large values of c we therefore expect that orbits will lie in the vicinity of this manifold. Since the projections of the manifold S onto the coordinates (x_1, y_1, x_2), (y_1, z_1, z_2) and (x_1, z_1, z_2) are paraboloids, we can actually see how this synchronization manifold is approached. This is illustrated in figure 8.3 in which the (x_1, y_1, x_2) projections of a typical trajectory are shown at four different coupling values.

Example 8.4 Consider a ring of n Lorenz systems coupled only through the x variable

$$\begin{aligned} x_i' &= \sigma(y_i - x_i) + d_x \Delta x_i \\ y_i' &= r x_i - y_i - x_i z_i \\ z_i' &= -\beta z_i + x_i y_i \end{aligned} \tag{40}$$

where $\Delta x_i = x_{i-1} - 2x_i + x_{i+1}$ with the index i taken modulo n is the discrete Laplacian coupling so that periodic boundary conditions are assumed. We define $\chi_{x_i} = x_{i+1 \bmod n} + x_i$ and $\eta_{x_i} = x_{i+1 \bmod n} - x_i$. In the limit $d_x \to \infty$, the manifold $S = \{\eta \in \mathbb{R}^n : \eta_{x_i} =$

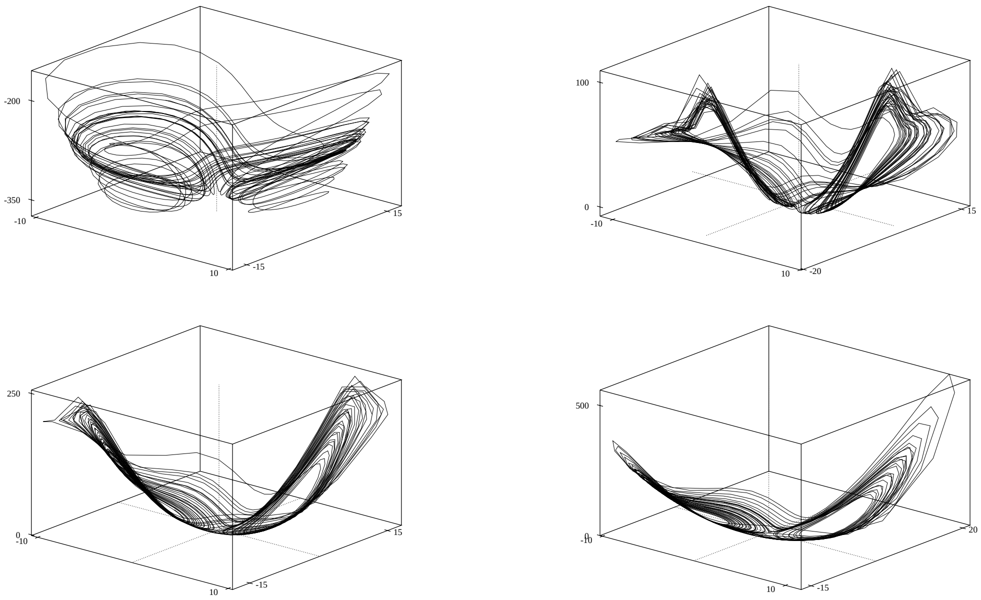

Figure 3: (x_1, y_1, x_2) projections of typical trajectories of the system (8.3) after transients die out. The four graphs correspond to coupling strengths of $k = 1$, $k = 50$, $k = 200$, and $k = 10000$ respectively.

0 for all $1 \leq i \leq n\}$ is a normally hyperbolic invariant manifold, as shown in [32]. Due to the special form of the Lorenz equations, the slow the evolution of $\eta_{y_i} = y_{i+1 \bmod n} - y_i$ and $\eta_{z_i} = z_{i+1 \bmod n} - z_i$ is governed by the linear equation

$$\begin{bmatrix} \dot{\eta}_{y_i} \\ \dot{\eta}_{z_i} \end{bmatrix} = \begin{bmatrix} -1 & x(t) \\ -x(t) & -\beta \end{bmatrix} \begin{bmatrix} \eta_{y_i} \\ \eta_{z_i} \end{bmatrix} = A(t) \begin{bmatrix} \eta_{y_i} \\ \eta_{z_i} \end{bmatrix}.$$

Since the eigenvalues of the matrix

$$\frac{1}{2}[A^{\mathrm{T}}(t) + A(t)] = \begin{bmatrix} -1 & 0 \\ 0 & -\beta \end{bmatrix} \tag{41}$$

are both negative, it follows from a simple result about the stability of non-autonomous linear systems [33, p. 307], that the origin is an asymptotically stable fixed point of equation (41). It can be checked using Lyapunov functions that there is an inflowing subset of the synchronization manifold for the slow equations, and thus the finite ring (40) will synchronize for sufficiently large values of the coupling d_x.

For large, but finite values of d_x, the manifold S remains a stable, normally hyperbolic, invariant manifold due to the symmetry of the equations. Therefore $|x_i - x_j|$ will decrease exponentially for all i and j. η_{y_i} and η_{z_i} must satisfy the following equation

$$\begin{bmatrix} \dot{\eta_{y_i}} \\ \dot{\eta_{z_i}} \end{bmatrix} = \begin{bmatrix} -1 & x(t) \\ -x(t) & -\beta \end{bmatrix} \begin{bmatrix} \eta_{y_i} \\ \eta_{z_i} \end{bmatrix} + \begin{bmatrix} (r + z_{i+1})\eta_{x_i} \\ y_{i+1}\eta_{x_i} \end{bmatrix}. \tag{42}$$

A direct application of Duhamel's formula now shows that the origin is an exponentially stable fixed point of these equations when η_{x_i} decays exponentially, and y_{i+1} and z_{i+1} remain bounded. It follows that for $d_x \gg 1$ all systems in the ring (40) will synchronize identically.

9 Limitations of the Description of Synchronization in Terms of Invariant Manifolds

It is possible that two systems in a network behave synchronously without the existence of a diffeomorphism from an open neighborhood of the attractor of one system to that of the other. In this case the description synchronization in terms of invariant manifolds is not applicable. In this section several such occurrences will be discussed.

Consider again the network (16). Generally we can define synchronization as the existence of differentiable functions f_i and f_j defined on the phase space of systems x_i and x_j respectively, such that

$$|f_i(x_i(t)) - f_j(x_j)(t)| < \epsilon \qquad t > 0$$

The functions f_i, and f_j could be identity maps in the case of nearly homogeneous network. More generally they could represent other physically relevant quantities such as the phase. In this situation there does not necessarily exist a synchronization manifold in the phase space of the network. Afraimovich, et. al. [27], [25], [28] have developed analytical techniques to show that for large couplings diffusively coupled inhomogeneous lattices of oscillators are nearly identically synchronized. These techniques are independent of the existence of invariant manifolds, and can thus be used to describe more general instances of synchronization. Note however that in section 8 it was shown that in the case of strong diffusive coupling in an inhomogeneous network the existence of a synchronization manifold can frequently be expected. It is not clear in what cases the network will be nearly synchronized without the existence of a synchronization manifold.

Perhaps even more important are the cases of phase locking and lag synchronization between chaotic phase-coherent systems. An approximate frequency ω and phase θ can be defined naturally on the attractor of many systems. Surprisingly, such systems frequently respond to a small periodic driving in a way very similar to periodically forced nonlinear phase oscillators. Let ω_0 and $\phi(t)$ be the frequency and phase of the periodic forcing. If ω_0 is close to ω and the forcing is sufficiently strong, then $|\theta(t) - \phi(t)| < \epsilon$ after transients die out[1].

There clearly exists a function from the orbits of one system to that of the other in the case of lag synchronization $x_2(t) = D(x_1(t)) = x_1(t - \delta)$. Elementary results about the differentiable dependence on initial conditions imply that D is a diffeomorphism, and its

[1]The origin of the phase θ is assume to be shifted so that there is no constant phase difference between the two oscillators. In general this expression would be $|\theta_0 + \theta(t) - \phi(t)| < \epsilon$.

graph is the synchronization manifold. As δ approaches 0, this synchronization manifold approaches the diagonal.

Remark 9.1 Both phase and lag synchronization are of great importance in biological applications. Neurons are frequently modeled as nonlinear phase oscillators, or by relaxation-oscillation equations. Experiments suggest that biological neurons are not exactly periodic. This non-periodicity is partly due to the random noise inherent in any physical systems, but some data analyses also suggest that the cause is deterministic [29]. The problem of phase locking and lag synchronization between systems with nonperiodic phase-coherent attractors is of direct relevance to this situation.

Unlike the case of phase oscillators, two coupled maps with hyperbolic compact attractors Λ_1 and Λ_2 neither of which is a fixed point, cannot be synchronized with an arbitrarily small C^1 coupling. This is due to the fact that $\Lambda_1 \times \Lambda_2$ is a hyperbolic attractor for the direct product of the two uncoupled system. A small coupling can be viewed as a small perturbation of this product, and will hence have an attractor on which the dynamics are topologically conjugate to the dynamics of the uncoupled system on $\Lambda_1 \times \Lambda_2$.

The situation is somewhat different in the case of flows. Even in the case of two hyperbolic systems, their direct product is no longer hyperbolic since it has *two* center directions. Notice that this is exactly the reason why two systems with attracting periodic orbits will generically be phase locked for small couplings. The case of chaotic attractors is significantly different from this example, since there will be n unstable direction at ρ-almost all points on the attractor, where ρ is an invariant measure, and n is the number of positive Lyapunov exponents. Therefore the attractor of the direct product of the flows has $2n$ unstable directions, while the attractor of the identically synchronous state has n unstable directions at each point. In general, we don't expect the drop from $2n$ unstable directions to n unstable directions to occur for arbitrarily small perturbations of the system.

10 Conclusion

It was shown that the description of chaotic synchronization in terms of invariant manifolds provides a natural framework for the study of many types of synchronization which were considered separate and distinct in the physics literature. In particular it is shown that, from an abstract point of view, there is no essential difference between generalized and identical synchronization, or synchronization between unidirectionally and bidirectionally coupled systems.

Once the theory of invariant manifolds was introduced, it was also possible to give precise mathematical meaning to quantities such as transversal Lyapunov exponents used in the physical literature. Moreover, necessary and sufficient conditions for the persistence of synchronization under perturbations follow as direct consequences of theorems that provide conditions for the persistence of invariant manifolds.

Three different methods for the detection of synchronization were introduced. Each is applicable to a particular class of systems. If the coupling in a network of systems has certain symmetries, then the synchronization manifold can often be found as a fixed set of a particular symmetry group acting on the phase space of the network. In case the network may be split into a drive and response system, the synchronization manifold may be found as an invariant section of an extension. Finally, in many cases of diffusively coupled systems, geometric singular perturbation theory may be used to establish the existence of a synchronization manifold.

Although mostly flows on $\mathbb{R}^n$ were considered, the theory of invariant manifolds carries over directly to maps and flows on arbitrary manifolds and all results discussed can be extended to maps and flows on abstract manifolds.

There is strong evidence that some interesting and important examples of chaotic synchronization can not be treated using this approach. It is possible that two systems are only approximately synchronized, or that their phase variables are approximately synchronized, without the existence of a synchronization manifold in the phase space. Some analytical tools have been developed to investigate such instances, however in general it is not even clear what definitions of chaotic synchronization are appropriate in such cases. One possible measure of the amount of synchronization between the systems could be an appropriate dimension of the attractor or an invariant measure on the attractor. It is expected that the dimension of the attractor in a network of n identical systems gradually decreases from $n \dim A$ to $\dim A$. A significant drop in the dimension could then be interpreted as an indication that some form of synchrony is present in the network.

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