PART I. THE REAL NUMBERS

This material assumes that you are already familiar with the real number system and the real line.

I.1. THE NATURAL NUMBERS AND INDUCTION

Let $\mathcal{N}$ denote the set of natural numbers (positive integers).

Axiom: If S is a nonempty subset of $\mathcal{N}$, then S has a least element. That is, there exists $m \in S$ such that $m \leq n$ for all $n \in S$. The natural numbers are *well-ordered.*

Mathematical Induction. Let S be a subset of $\mathcal{N}$. If S has the following properties:

1. $1 \in S$, and
2. $k \in S$ implies $k+1 \in S$,

then $S = \mathcal{N}$.

Proof: Suppose $S \neq \mathcal{N}$. Let $T = \mathcal{N} - \mathcal{S}$. Then $T \neq \emptyset$. Let m be the least element in T. Then $m-1 \notin T$. Therefore, $m-1 \in S$ which implies that $(m-1)+1 = m \in S$ (by 2), a contradiction.

Corollary: Let S be a subset of $\mathcal{N}$ such that

1. $m \in S$.
2. If $k \geq m \in S$, then $k+1 \in S$.

Then, $S = \{n \in \mathcal{N} : n \geq m\}$.

Examples

Exercises 1.1

1. Prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.
2. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
3. Let r be a real number $r \neq 1$. Prove that
$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$
4. Prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$.
5. True or False: If S is a non-empty subset of $\mathcal{N}$, then there exists an element $m \in S$ such that $m \geq k$ for all $k \in S$.

I.2. ORDERED FIELDS

Let $\mathbb{R}$ denote the set of real numbers. The set $\mathbb{R}$, together with the operations of addition $(+)$ and multiplication $(\cdot)$, satisfies the following axioms:

Addition:

A1. For all $x,\ y \in \mathbb{R},\ x + y \in \mathbb{R}$ (addition is a closed operation).

A2. For all $x,\ y \in \mathbb{R},\ x + y = y + x$ (addition is commutative)

A3. For all $x,\ y,\ z \in \mathbb{R},\ x + (y + z) = (x + y) + z$ (addition is associative).

A4. There is a unique number 0 such that $x + 0 = 0 + x$ for all $x \in \mathbb{R}$. (0 is the *additive identity.*)

A5. For each $x \in \mathbb{R}$, there is a unique number $-x \in \mathbb{R}$ such that $x + (-x) = 0$. ($-x$ is the *additive inverse of* x.)

Multiplication:

M1. For all $x,\ y \in \mathbb{R},\ x \cdot y \in \mathbb{R}$ (multiplication is a closed operation).

M2. For all $x,\ y \in \mathbb{R},\ x \cdot y = y \cdot x$ (multiplication is commutative)

M3. For all $x,\ y,\ z \in \mathbb{R},\ x \cdot (y \cdot z) = (x \cdot y) \cdot$ (multiplication is associative).

M4. There is a unique number 1 such that $x \cdot 1 = 1 \cdot x$ for all $x \in \mathbb{R}$. (1 is the *multiplicative identity.*)

M5. For each $x \in \mathbb{R},\ x \neq 0$, there is a unique number $1/x = x^{-1} \in \mathbb{R}$ such that $x \cdot (1/x) = 1$. ($1/x$ is the *multiplicative inverse of* x.)

Distributive Law:

D. For all $x,\ y,\ z \in \mathbb{R},\ x \cdot (y + z) = x \cdot y + x \cdot z$.

A non-empty set S together with two operations, "addition" and "multiplication" which satisfies A1-A5, M1-M5, and D is called a *field*. The set of real numbers with ordinary addition and multiplication is an example of a field.

Order:

There is a subset P of $\mathbb{R}$ that has the following properties:

a If $x,\ y \in P$, then $x + y \in P$.

b If $x,\ y \in P$, then $x \cdot y \in P$.

c For each $x \in \mathbb{R}$ exactly one of the following holds: $x \in P$, $x = 0$, $-x \in P$.

The set P is known as the set of positive numbers.

Let $x,\ y \in \mathbb{R}$. Then $x < y$ (read "x is less than y") if $y - x \in P$. $x < y$ is equivalent to $y > x$ (read "y is greater than x"). $P = \{x \in \mathbb{R} : x > 0\}$. $x \leq y$ means either $x < y$ or $x = y$; $y \geq x$ means either $y > x$ or $y = x$.

The relation "$<$" has the following properties:

O1. For all $x,\ y \in \mathbb{R}$, exactly one of the following holds: $x < y,\ x = y,\ x > y$. (Trichotomy Law)

O2. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$ and $x + z < y + z$.

O4. For all $x,\ y,\ z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$.

$\{\mathbb{R}, +, \cdot, <\}$ is an *ordered field*. *Any* mathematical system $\{S, +, \cdot, <\}$ satisfying these 15 axioms is an ordered field. In particular, the set of rational numbers $\mathcal{Q}$, together with ordinary addition, multiplication and "less than", is an ordered field, a *subfield of* $\mathbb{R}$.

THEOREM 1. *Let $x,\ y \in \mathbb{R}$. If $x \leq y + \epsilon$ for every positive number ϵ, then $x \leq y$.*

Definition 1. *Let $x \in \mathbb{R}$. The absolute value of x, denoted $|x|$, is given by*

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The properties of absolute value are: for any $x,\ y \in \mathbb{R}$.

(1) $|x| \geq 0$,

(2) $|xy| = |x|\,|y|$,

(3) $|x + y| \leq |x| + |y|$.

Exercises 1.2

1. True − False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) If $x,\ y,\ z \in \mathbb{R}$ and $x < y$, then $xz < yz$.
   (b) If $x,\ y \in \mathbb{R}$ and $x < y + \epsilon$ for every positive number ϵ, then $x < y$.
   (c) If $x,\ y \in \mathbb{R}$, then $|x - y| \leq |x| + |y|$.
   (d) If $x,\ y \in \mathbb{R}$, then $|\,|x| - |y|\,| \leq |x| - |y|$.
2. Prove: If $|x - y| < \epsilon$ for every $\epsilon > 0$, then $x = y$.

3. Suppose that $x_1,\ x_2,\ \ldots,\ x_n$ are real numbers. Prove that

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$

I.3. THE COMPLETENESS AXIOM

$\mathbb{R}$ and $\mathcal{Q}$ are each ordered fields. What distinguishes $\mathbb{R}$ from $\mathcal{Q}$ is the completeness axiom. As you know, $\mathcal{Q}$ is a proper subset of $\mathbb{R}$; i.e., there are real numbers which are not rational numbers. Such numbers are called *irrational numbers*.

THEOREM 2. $\sqrt{2}$ *is not a rational number. In general, if* p *is a prime number, then* $\sqrt{p}$ *is not a rational number.*

Other examples of irrational numbers are π and e.

Definition 2. *Let* S *be a subset of* $\mathbb{R}$*. A number* $u \in \mathbb{R}$ *is an* **upper bound** *of* S *if* $s \le u$ *for all* $s \in S$*. An element* $w \in \mathbb{R}$ *is a* **lower bound** *of* S *if* $w \le s$ *for all* $s \in S$*. If an upper bound* u *for* S *is an element of* S*, then* u *is called the* **maximum** *(or* **largest element***) of* S*. Similarly, if a lower bound* w *for* S *is an element of* S*, then* w *is called the* **minimum** *(or* **smallest element***) of* S*.*

Examples: If $S = (0, 1)$, then 2 is an upper bound for S and -5 is a lower bound for S.

Definition 3. *A set* $S \subseteq \mathbb{R}$ *is said to be* **bounded above** *if* S *has an upper bound;* S *is* **bounded below** *if it has a lower bound. A subset* S *of* $\mathbb{R}$ *is* **bounded** *if it has both an upper bound and a lower bound.*

Definition 4. *Let* $S \subseteq \mathbb{R}$ *be a set that is bounded above. A number* $u \in \mathbb{R}$ *is called the supremum (least upper bound) of* S *denoted by* $\sup S$ *if it satisfies the conditions*

1. $s \le u$ *for all* $s \in S$.
2. *If* v *is an upper bound for* S, *then* $u \le v$.

THEOREM 3. *Let* $S \subseteq \mathbb{R}$ *and suppose that* $u = \sup S$*. Then, given any positive number* ϵ*, there is an element* $s_\epsilon \in S$ *such that* $u - \epsilon < s_\epsilon \le u$*.*

Definition 5. *: Let* $S \subseteq \mathbb{R}$ *be a set that is bounded below. A number* $u \in \mathbb{R}$ *is called the infimum (greatest lower bound) of* S *and is denoted by* $\inf S$ *if it satisfies the conditions*

1. $u \le s$ *for all* $s \in S$.
2. *If* v *is a lower bound for* S, *then* $v \le u$.

The Completeness Axiom Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

The set of real numbers $\mathbb{R}$ is a complete, ordered, field. The set of rational numbers $\mathcal{Q}$, although an ordered field, is not complete. For example, the set $T = \{r \in \mathcal{Q} : r < \sqrt{2}\}$ is bounded above, but T does not have a rational least upper bound.

The Archimedean Property

THEOREM 4. *(The Archimedean Property) The set $\mathcal{N}$ of natural numbers is unbounded above.*

THEOREM 5. *The following are equivalent:*

(a) The Archimedean Property.

(b) For each $z \in \mathbb{R}$, there exists an $n \in \mathcal{N}$ such that $n > z$.

(c) For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathcal{N}$ such that $nx > y$.

(d) For each $x > 0$, there exists an $n \in \mathcal{N}$ such that $0 < 1/n < x$.

THEOREM 6. *There exists a real number x such that $x^2 = 2$. In general, if p is a prime number, then there exists a real number y such that $y^2 = p$.*

The Density of the Rational Numbers

THEOREM 7. *If x and y are real numbers, $x < y$, then there exists a rational number r such that $x < r < y$.*

THEOREM 8. *If x and y are real numbers, $x < y$, then there exists an irrational number z such that $x < z < y$.*

Exercises 1.3

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) If a non-empty subset of $\mathbb{R}$ has a infimum, then it is bounded.

   (b) Every non-empty bounded subset of $\mathbb{R}$ has a maximum and a minimum.

   (c) If v is an upper bound for S $u < v$, then u is not an upper bound for S.

   (d) If $w = \inf S$ and $z < w$, then z is a lower bound for S.

   (e) Every nonempty subset of $\mathcal{N}$ has a minimum.

   (f) Every nonempty subset of $\mathcal{N}$ has a maximum.

2. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) If x and y are irrational, then xy is irrational.

   (b) Between any two distinct rational numbers, there is an irrational number.

(c) Between any two distinct irrational numbers, there is a rational number.

(d) The rational and irrational numbers alternate.

3. Let $S \subseteq \mathbb{R}$ be non-empty and bounded above and let $u = \sup S$. Prove that $u \in S$ if and only if $u = \max S$.

4. (a) Let $S \subseteq \mathbb{R}$ be non-empty and bounded above and let $u = \sup S$. Prove that u is unique.

   (b) Prove that if each of m and n is a maximum of S, then $m = n$.

5. Let $S \subseteq \mathbb{R}$ and suppose that $v = \inf S$. Then, given any positive number ϵ, there is an element $s_\epsilon \in S$ such that $v \leq s_\epsilon < v + \epsilon$.

6. Prove that if x and y are real numbers with $x < y$, then there are infinitely rational numbers in the interval $[x, y]$.

I.4. TOPOLOGY OF THE REALS

Definition 6. *Let $S \subseteq \mathbb{R}$. The set $S^c = \{x \in \mathbb{R} : x \notin S\}$ is called the* **complement** *of S.*

Definition 7. *Let $x \in \mathbb{R}$ and let $\epsilon > 0$. An* **ϵ-neighborhood** *of x (often shortened to "a neighborhood of x") is the set*

$$N(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\}.$$

The number ϵ is called the radius of $N(x, \epsilon)$.

Note that an ϵ-neighborhood of a point x is the open interval $(x - \epsilon, x + \epsilon)$.

Definition 8. *Let $x \in \mathbb{R}$ and let $\epsilon > 0$. A* **deleted ϵ-neighborhood** *of x (often shortened to "a deleted neighborhood of x") is the set*

$$N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}.$$

A deleted ϵ-neighborhood of x is an ϵ-neighborhood of x with the point x removed;

$$N^*(x, \epsilon) = (x - \epsilon, x) \cup (x, x + \epsilon).$$

Definition 9. *Let $S \subseteq \mathbb{R}$. A point $x \in S$ is an* **interior point** *of S if there exists a neighborhood N of x such that $N \subseteq S$. The set of all interior points of S is denoted by $\text{int}\, S$.*

Examples

Definition 10. *Let $S \subseteq \mathbb{R}$. A point $z \in S$ is a* **boundary point** *of S if $N \cap S \neq \emptyset$ and $N \cap S^c \neq \emptyset$ for every neighborhood N of z. The set of all boundary points of S is denoted by* bd S.

Examples

Open Sets and Closed Sets

Definition 11. *Let* $S \subseteq \mathbb{R}$. S *is* **open** *if every point of* S *is an interior point. That is,* S *is open if and only if* $S = \operatorname{int} S$. S *is* **closed** *if and only if* S^c *is open.*

Examples

THEOREM 9. *Let* $S \subseteq \mathbb{R}$. S *is closed if and only if* $\text{bd } S \subseteq S$.

THEOREM 10. *(a) The union of any collection of open sets is open.*

(b) The intersection of any finite collection of open sets is open.

Corollary 1. *(a) The intersection of any collection of closed sets is closed.*

(b) The union of any finite collection of closed sets is closed.

The Corollary follows directly from the Theorem by means of **De Morgan's Laws :**

Let A be a set of real numbers. Then

1. $\left(\cup_{\alpha \in A} S_\alpha\right)^c = \cap_{\alpha \in A} S_\alpha^c$
2. $\left(\cap_{\alpha \in A} S_\alpha\right)^c = \cup_{\alpha \in A} S_\alpha^c$

Examples

Accumulation Points

Definition 12. *Let* $S \subseteq \mathbb{R}$. *A point* $x \in \mathbb{R}$ *is an* **accumulation point** *of* S *if every deleted neighborhood* N *of* x *contains a point of* S. *The set of accumulation points of* S *is denoted by* S'. *If* $x \in S$ *and* x *is not an accumulation point of* S, *then* x *is an* **isolated point** *of* S.

Examples

Note, an accumulation point of S may or may not be a point of S.

Definition 13. *Let* $S \subseteq \mathbb{R}$. *The* **closure** *of* S, *denoted by* $\text{cl } S$, *is the set*

$$\text{cl } S = S \cup S'.$$

THEOREM 11. *Let* $S \subseteq \mathbb{R}$. *Then*

(a) S *is closed if and only if* S *contains all its accumulation points.*

(b) $\text{cl } S$ *is a closed set.*

Exercises 1.4

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.
   (a) int $S \cap$ bd $S = \emptyset$.
   (b) bd $S \subseteq$.
   (c) S is closed if and only if $S =$ bd S.
   (d) If $x \in S$, then $x \in$ int S or $x \in$ bd S.
   (e) bd $S =$ bd S^c.
   (f) bd $S \subseteq S^c$.
2. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.
   (a) A neighborhood is an open set.
   (b) The union of any collection of open sets is open.
   (c) The union of any collection of closed sets is closed.
   (d) The intersection of any collection of open sets is open.
   (e) The intersection of any collection of closed sets is closed.
   (f) $\mathbb{R}$ is neither open nor closed.
3. Classify each set as open, closed, neither, or both.
   (a) $\mathcal{N}$
   (b) $\mathcal{Q}$
   (c) $\left\{\frac{1}{n} : n \in \mathcal{N}\right\}$
   (d) $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$
   (e) $\{x : x^2 > 0\}$
   (f) $\{x : |x-2| \leq 3\}$
4. Let S be a bounded infinite set and let $u = \sup S$.
   (a) Prove that if $u \notin S$, then $u \in S'$.
   (b) True or false: if $u \in S$, then $u \notin S'$?
5. Prove that if x is an accumulation point of S, then every neighborhood of x contains infinitely many points of S.

I.5. COMPACT SETS

Definition 14. *Let $S \subseteq \mathbb{R}$. A collection $\mathcal{G}$ of open sets such that $S \subseteq \bigcup_{G \in \mathcal{G}} G$ is called an* **open cover** *of S. A subcollection $\mathcal{F}$ of $\mathcal{G}$ which also covers S is called a* **subcover** *of S.*

Examples

Definition 15. *A set* $S \subseteq \mathbb{R}$ *is* **compact** *if and only if every open cover* $\mathcal{G}$ *of* S *contains a finite subcover. That is,* S *is compact if for every open cover* $\mathcal{G}$ *of* S *there is a finite collection of open sets* $G_1,\ G_2, \ldots\ G_n$ *in* $\mathcal{G}$ *such that* $S \subseteq \bigcup_{k=1}^{n} G_k$.

Examples

THEOREM 12. *If* $S \subseteq \mathbb{R}$ *is non-empty, closed and bounded, then* S *has a maximum and a minimum.*

THEOREM 13. *(Heine-Borel Theorem) A subset* S *of* $\mathbb{R}$ *is compact if and only if it is closed and bounded.*

THEOREM 14. *(Bolzano-Weierstrass Theorem) If* $S \subseteq \mathbb{R}$ *is a bounded infinite set, then* S *has at least one accumulation point.*

Exercises 1.5

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) Every finite set is compact.
   (b) No infinite set is compact.
   (c) If a set is compact, then it has a maximum and a minimum.
   (d) If a set has a maximum and a minimum, then it is compact.
   (e) If $S \subseteq \mathbb{R}$ is compact, then there is at least one point $x \in \mathbb{R}$ such that x is an accumulation point of S.
   (f) If $S \subseteq \mathbb{R}$ is compact and x is an accumulation point of S, then $x \in S$.
2. Show that each of the following subsets S of $\mathbb{R}$ is not compact by giving an open cover of S that has no finite subcover.
   (a) $S = [0, 1)$
   (b) $S = \mathcal{N}$
   (c) $S = \{1/n : n \in \mathcal{N}\}$
3. Prove that the intersection of any collection of compact sets is compact.
4. Prove that if $S \subseteq \mathbb{R}$ is compact and T is a closed subset of S, then T is compact.

PART II. SEQUENCES OF REAL NUMBERS

II.1. CONVERGENCE

Definition 16. *A* **sequence** *is a real-valued function* f *whose domain is the set positive integers* ($\mathcal{N}$). *The numbers* $f(1),\ f(2),\ \cdots$ *are called the* **terms** *of the sequence.*

Notation Function notation vs subscript notation:

$$f(1) \equiv s_1,\ f(2) \equiv s_2, \cdots,\ f(n) \equiv s_n,\ \cdots .$$

Examples

Definition 17. *A sequence* $\{s_n\}$ **converges** *to the number* s *if to each* $\epsilon > 0$ *there corresponds a positive integer* N *such that*

$$|s_n - s| < \epsilon \quad \textit{for all } n > N.$$

The number s *is called the* **limit** *of the sequence.*

Notation "$\{s_n\}$ converges to s" is denoted by

$$\lim_{n\to\infty} s_n = s \quad \text{or by} \quad \lim s_n = s \quad \text{or by} \quad s_n \to s.$$

A sequence that does not converge is said to **diverge**.

Examples

THEOREM 15. *If* $s_n \to s$ *and* $s_n \to t$, *then* $s = t$. *That is, the limit of a convergent sequence is unique.*

THEOREM 16. *If* $\{s_n\}$ *converges, then* $\{s_n\}$ *is bounded.*

THEOREM 17. *Let* $\{s_n\}$ *and* $\{a_n\}$ *be sequences and suppose that there is a positive number* k *and a positive integer* N *such that*

$$|s_n| \le k\, a_n \quad \textit{for all } n > N.$$

If $a_n \to 0$, *then* $s_n \to 0$.

Corollary Let $\{t_n\}$ and $\{a_n\}$ be sequences and let $t \in \mathbb{R}$. Suppose that there is a positive number k and a positive integer N such that

$$|t_n - t| \le k\, a_n \quad \text{for all } n > N.$$

If $a_n \to 0$, then $t_n \to t$.

Exercises 2.1

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.

   (a) If $s_n \to s$, then $s_{n+1} \to s$.

   (b) If $s_n \to s$ and $t_n \to s$, then there is a positive integer N such that $s_n = t_n$ for all $n > N$.

   (c) Every bounded sequence converges

   (d) If to each $\epsilon > 0$ there is a positive integer N such that $n > N$ implies $s_n < \epsilon$, then $s_n \to 0$.

   (e) If $s_n \to s$, then s is an accumulation point of the set $S = \{s_1, s_2, \cdots\}$.

2. Prove that $\lim \dfrac{3n+1}{n+2} = 3$.

3. Prove that $\lim \dfrac{\sin n}{n} = 0$.

4. Prove or give a counterexample:

   (a) If $\{s_n\}$ converges, then $\{|s_n|\}$ converges.

   (b) If $\{|s_n|\}$ converges, then $\{s_n\}$ converges.

5. Give an example of:

   (a) A convergent sequence of rational numbers having an irrational limit.

   (b) A convergent sequence of irrational numbers having a rational limit.

II.2. LIMIT THEOREMS

THEOREM 18. *Suppose $s_n \to s$ and $t_n \to t$. Then:*

1. $s_n + t_n \to s + t$.

2. $s_n - t_n \to s - t$.

3. $s_n t_n \to st$.

   Special case: $ks_n \to ks$ for any number k.

4. $s_n/t_n \to s/t$ *provided* $t \neq 0$ *and* $t_n \neq 0$ *for all* n.

THEOREM 19. *Suppose $s_n \to s$ and $t_n \to t$. If $s_n \leq t_n$ for all n, then $s \leq t$.*

Corollary Suppose $t_n \to t$. If $t_n \geq 0$ for all n, then $t \geq 0$.

Infinite Limits

Definition 18. *A sequence* $\{s_n\}$ **diverges to** $+\infty$ $(s_n \to +\infty)$ *if to each real number* M *there is a positive integer* N *such that* $s_n > M$ *for all* $n > N$. $\{s_n\}$ **diverges to** $-\infty$ $(s_n \to -\infty)$ *if to each real number* M *there is a positive integer* N *such that* $s_n < M$ *for all* $n > N$.

THEOREM 20. *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are sequences such that* $s_n \le t_n$ *for all* n.

1. *If* $s_n \to +\infty$, *then* $t_n \to +\infty$.
2. *If* $t_n \to -\infty$, *then* $s_n \to -\infty$.

THEOREM 21. *Let* $\{s_n\}$ *be a sequence of positive numbers. Then* $s_n \to +\infty$ *if and only if* $1/s_n \to 0$.

Exercises 2.2

1. Prove or give a counterexample.

   (a) If $s_n \to s$ and $s_n > 0$ for all n, then $s > 0$.

   (b) If $\{s_n\}$ and $\{t_n\}$ are divergent sequences, then $\{s_n + t_n\}$ is divergent.

   (c) If $\{s_n\}$ and $\{t_n\}$ are divergent sequences, then $\{s_n t_n\}$ is divergent.

   (d) If $\{s_n\}$ and $\{s_n + t_n\}$ are convergent sequences, then $\{t_n\}$ is convergent.

   (e) If $\{s_n\}$ and $\{s_n t_n\}$ are convergent sequences, then $\{t_n\}$ is convergent.

2. Determine the convergence or divergence of $\{s_n\}$. Find any limits that exist.

   (a) $s_n = \dfrac{3-2n}{1+n}$ (b) $s_n = \dfrac{(-1)^n}{n+2}$

   (c) $s_n = \dfrac{(-1)^n n}{2n-1}$ (d) $s_n = \dfrac{2^{3n}}{3^{2n}}$

   (e) $s_n = \dfrac{n^2-2}{n+1}$ (f) $s_n = \dfrac{1+n+n^2}{1+3n}$

II.3. MONOTONE SEQUENCES AND CAUCHY SEQUENCES

Monotone Sequences

Definition 19. *A sequence* $\{s_n\}$ *is* **increasing** *if* $s_n \le s_{n+1}$ *for all* n; $\{s_n\}$ *is* **decreasing** *if* $s_n \ge s_{n+1}$ *for all* n. *A sequence is* **monotone** *if it is increasing or if it is decreasing.*

Examples

THEOREM 22. *A monotone sequence in convergent if and only if it is bounded.*

THEOREM 23. *(a) If* $\{s_n\}$ *is increasing and unbounded, then* $s_n \to +\infty$.

(b) If $\{s_n\}$ *is decreasing and unbounded, then* $s_n \to -\infty$.

Cauchy Sequences

Definition 20. *A sequence* $\{s_n\}$ *is a* **Cauchy sequence** *if to each* $\epsilon > 0$ *there is a positive integer* N *such that*

$$m,\ n > N \quad \textit{implies} \quad |s_n - s_m| < \epsilon.$$

THEOREM 24. *Every convergent sequence is a Cauchy sequence.*

THEOREM 25. *Every Cauchy sequence is bounded.*

THEOREM 26. *A sequence* $\{s_n\}$ *is convergent if and only if it is a Cauchy sequence.*

Exercises 2.3

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) If a monotone sequence is bounded, then it is convergent.
   (b) If a bounded sequence is monotone, then it is convergent.
   (c) If a convergent sequence is monotone, then it is bounded.
   (d) If a convergent sequence is bounded, then it is monotone.
2. Give an example of a sequence having the given properties.
   (a) Cauchy, but not monotone.
   (b) Monotone, but not Cauchy.
   (c) Bounded, but not Cauchy.
3. Show that the sequence $\{s_n\}$ defined by $s_1 = 1$ and $s_{n+1} = \frac{1}{4}(s_n + 5)$ is monotone and bounded. Find the limit.
4. Show that the sequence $\{s_n\}$ defined by $s_1 = 2$ and $s_{n+1} = \sqrt{2s_n + 1}$ is monotone and bounded. Find the limit.

II.4. SUBSEQUENCES

Definition 21. *Given a sequence* $\{s_n\}$*. Let* $\{n_k\}$ *be a sequence of positive integers such that* $n_1 < n_2 < n_3 < \cdots$ *. The sequence* $\{s_{n_k}\}$ *is called a* **subsequence** *of* $\{s_n\}$.

Examples

THEOREM 27. *If* $\{s_n\}$ *converges to* s, *then every subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *also converges to* s.

Corollary If $\{s_n\}$ has a subsequence $\{t_n\}$ that converges to α and a subsequence $\{u_n\}$ that converges to β with $\alpha \neq \beta$, then $\{s_n\}$ does not converge.

THEOREM 28. *Every bounded sequence has a convergent subsequence.*

THEOREM 29. *Every unbounded sequence has a monotone subsequence that diverges either to* $+\infty$ *or to* $-\infty$.

Limit Superior and Limit Inferior

Definition 22. *Let* $\{s_n\}$ *be a bounded sequence. A number* α *is a* **subsequential limit** *of* $\{s_n\}$ *if there is a subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *such that* $s_{n_k} \to \alpha$.

Examples

Let $\{s_n\}$ be a bounded sequence. Let

$$S = \{\alpha : \alpha \text{ is a subsequential limit of } \{s_n\}.$$

Then:

1. $S \neq \emptyset$.
2. S is a bounded set.

Definition 23. *Let* $\{s_n\}$ *be a bounded sequence and let* S *be its set of subsequential limits. The* **limit superior** *of* $\{s_n\}$ *(denoted by lim sup* s_n*) is*

$$\mathit{lim\ sup}\ s_n = \mathit{sup}\ S.$$

The **limit inferior** *of* $\{s_n\}$ *(denoted by lim inf* s_n*) is*

$$\mathit{lim\ inf}\ s_n = \mathit{inf}\ S.$$

Examples

Clearly, $\liminf s_n \leq \limsup s_n$.

Definition 24. *Let* $\{s_n\}$ *be a bounded sequence.* $\{s_n\}$ **oscillates** *if* $\mathit{lim\ inf}\ s_n < \mathit{lim\ sup}\ s_n$.

Exercises 2.4

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

(a) A sequence $\{s_n\}$ converges to s if and only if every subsequence of $\{s_n\}$ converges to s.

(b) Every bounded sequence is convergent.

(c) Let $\{s_n\}$ be a bounded sequence. If $\{s_n\}$ oscillates, then the set S of subsequential limits of $\{s_n\}$ has at least two points.

(d) Every sequence has a convergent subsequence.

(e) $\{s_n\}$ converges to s if and only if $\liminf s_n = \limsup s_n = s$.

2. Prove or give a counterexample.

(a) Every oscillating sequence has a convergent subsequence.

(b) Every oscillating sequence diverges.

(c) Every divergent sequence oscillates.

(d) Every bounded sequence has a Cauchy subsequence.

(e) Every monotone sequence has a bounded subsequence.

PART III. FUNCTIONS: LIMITS AND CONTINUITY

III.1. LIMITS OF FUNCTIONS

This part is concerned with functions $f : D \to \mathbb{R}$ where D is a nonempty subset of $\mathbb{R}$. That is, we will be considering real-valued functions of a real variable. The set D is called the *domain* of f.

Definition 25. *Let* $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D. *A number* L *is the* **limit of** f **at** c *if to each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad 0 < |x - c| < \delta.$$

This definition can be stated equivalently as follows:

Definition. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the **limit of** f **at** c if to each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.

Notation $\lim\limits_{x \to c} f(x) = L.$

Examples

THEOREM 30. *Let* $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D. *Then* $\lim\limits_{x \to c} f(x) = L$ *if and only if for every sequence* $\{s_n\}$ *in* D *such that* $s_n \to c$, $s_n \neq c$ *for all* n, $f(s_n) \to L$.

Corollary Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. If $\lim\limits_{x \to c} f(x)$ exists, then it is unique. That is, f can have only one limit at c.

THEOREM 31. *Let* $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D. *The following are equivalent:*

1. *f does not have a limit at c.*
2. *There exists a sequence $\{s_n\}$ in D such that $s_n \to c$, but $\{f(s_n)\}$ does not converge.*

Arithmetic of Limits

THEOREM 32. *Let* $f, g : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D. *If*

$$\lim_{x \to c} f(x) = L \quad \textit{and} \quad \lim_{x \to c} g(x) = M,$$

then

1. $\lim\limits_{x \to c} [f(x) + g(x)] = L + M$,
2. $\lim\limits_{x \to c} [f(x) - g(x)] = L - M$,

3. $\lim_{x\to c}[f(x)g(x)] = LM, \quad \lim_{x\to c}[k\,f(x)] = kL,$ k *constant,*

4. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ *provided* $M \neq 0,\ g(x) \neq 0.$

Examples

THEOREM 33. *(**"Pinching Theorem"**) Let* $f,\ g,\ h : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D*. Suppose that* $f(x) \le g(x) \le h(x)$ *for all* $x \in D,\ x \neq c$*. If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then $\lim_{x\to c} g(x) = L.$

One-Sided Limits

Definition 26. *Let* $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D*. A number* L *is the* **right-hand limit of** f **at** c *if to each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad c < x < c + \delta.$$

Notation: $\lim_{x\to c^+} f(x) = L.$

A number M *is the* **left-hand limit of** f **at** c *if to each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad c - \delta < x < c.$$

Notation: $\lim_{x\to c^-} f(x) = M.$

Examples

THEOREM 34. $\lim_{x\to c} f(x) = L$ *if and only if each of the one-sided limits* $\lim_{x\to c^+} f(x)$ *and* $\lim_{x\to c^-} f(x)$ *exists, and*

$$\lim_{x\to c+} f(x) = \lim_{x\to c^-} f(x) = L.$$

Exercises 3.1

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) $\lim_{x\to c} f(x) = L$ if and only if to each $\epsilon > 0$, there is a $\delta > 0$ such that

   $$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta, \quad x \in D.$$

   (b) $\lim_{x\to c} f(x) = L$ if and only if for each deleted neighborhood U of c there is a neighborhood V of L such that $f(U \cap D) \subseteq V$.

   (c) $\lim_{x\to c} f(x) = L$ if and only if for every sequence $\{s_n\}$ in D that converges to c, $s_n \neq c$ for all n, the sequence $\{f(s_n)\}$ converges to L.

(d) If f does not have a limit at c, then there exists a sequence $\{s_n\}$ in D $s_n \neq c$ for all n, such that $s_n \to c$, but $\{f(s_n)\}$ diverges.

(e) For any polynomial P and any real number c, $\lim_{x \to c} P(x) = P(c)$.

(f) For any polynomials P and Q, and any real number c,

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

2. Find a $\delta > 0$ such that $|x - 3| < \delta$ implies $|x^2 - 5x + 6| < \frac{1}{4}$.

3. Find a $\delta > 0$ such that $|x - 2| < \delta$ implies $|x^2 + 2x - 8| < \frac{1}{10}$.

4. Prove that $\lim_{x \to 3} (x^2 - 2x + 3) = 6$.

5. Determine whether or not the following limits exist:

   (a) $\lim_{x \to 0} \left|\sin \frac{1}{x}\right|$.

   (b) $\lim_{x \to 0} x \sin \frac{1}{x}$.

6. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that $\lim_{x \to c} f(x) = L$ and $L > 0$. Prove that there is a number $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ with $0 < |x - c| < \delta$.

III.2 CONTINUOUS FUNCTIONS

Definition 27. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f* **is continuous at** *c if to each $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta, \quad x \in D.$$

Let $S \subseteq D$. Then f is continuous on S if it is continuous at each point $c \in S$. f is continuous if f is continuous on D.

Examples

THEOREM 35. Characterizations of Continuity *Let $f : D \to \mathbb{R}$ and let $c \in D$. The following are equivalent:*

1. *f is continuous at c.*
2. *If $\{x_n\}$ is a sequence in D such that $x_n \to c$, then $f(x_n) \to f(c)$.*
3. *To each neighborhood V of $f(c)$, there is a neighborhood U of c such that $f(U \cap D) \subseteq V$.*

Corollary *If c is an accumulation point of D, then each of the above is equivalent to*

$$\lim_{x \to c} f(x) = f(c).$$

THEOREM 36. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c if and only if there is a sequence $\{x_n\}$ in D such that $x_n \to c$ but $\{f(x_n)\}$ does not converge to $f(c)$.*

Examples

Continuity of Combinations of Functions

THEOREM 37. Arithmetic: *Let $f,\ g : D \to \mathbb{R}$ and let $c \in D$. If f and g are continuous at c, then*

1. *$f + g$ is continuous at c.*
2. *$f - g$ is continuous at c.*
3. *fg is continuous at c; kf is continuous at c for any constant k.*
4. *f/g is continuous at c provided $g(c) \neq 0$.*

THEOREM 38. Composition: *Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at $c \in D$ and g is continuous at $f(c) \in E$, then the composition of g with f, $g \circ f : D \to \mathbb{R}$, is continuous at c.*

Examples

Definition 28. *Let $f : D \to \mathbb{R}$, and let $G \subseteq \mathbb{R}$. The pre-image of G, denoted by $f^{-1}(G)$ is the set*

$$f^{-1}(G) = \{x \in D : f(x) \in G\}.$$

THEOREM 39. *A function $f : D \to \mathbb{R}$ is continuous on D if and only if for each open set G in $\mathbb{R}$ there is an open set H in $\mathbb{R}$ such that $H \cap D = f^{-1}(G)$.*

Corollary A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

Exercises 3.2

1. Let $f : D \to \mathbb{R}$ and let $c \in D$. True – False. Justify your answer by citing a definition or theorem, giving a proof, or giving a counter-example.

   (a) f is continuous at c if and only if to each ϵ there is a $\delta > 0$ such that

   $$|fx) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta \text{ and } x \in D.$$

   (b) If $f(D) \subseteq \mathbb{R}$ is bounded, then f is continuous on D.

   (c) If c is an isolated point of D, then f is continuous at c.

   (d) If f is continuous at c and $\{x_n\}$ is a sequence in D, then $x_n \to c$ whenever $f(x_n) \to f(c)$.

(e) If $\{x_n\}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is convergent.

2. Let $f(x) = \dfrac{x^2+2x-15}{x-3}$. Define f at 3 so that f will be continuous at 3.

3. Prove or give a counterexample.

   (a) If f and $f+g$ are continuous on D, then g is continuous on D.

   (b) If f and fg are continuous on D, then g is continuous on D.

   (c) If f and g are not continuous on D, then $f+g$ is not continuous on D.

   (d) If f and g are not continuous on D, then fg is not continuous on D.

   (e) If f^2 is continuous on D, then f is continuous on D.

   (f) If f is continuous on D, then $f(D)$ is a bounded set.

4. Let $f : D \to \mathbb{R}$.

   (a) Prove that if f is continuous at c, then $|f|$ is continuous at c.

   (b) Suppose that $|f|$ is continuous at c. Does it follow that f is continuous at c? Justify your answer.

5. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that if $f(c) > 0$, then there is an $\alpha > 0$ and a neighborhood U of c such that $f(x) > \alpha$ for all $x \in U \cap D$.

6. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \le M$ for all $x \in U \cap D$.

III.3. PROPERTIES OF CONTINUOUS FUNCTIONS

Definition 29. *A function $f : D \to \mathbb{R}$ is* **bounded** *if there exists a number M such that $|f(x)| \le M$ for all $x \in D$. That is, f is bounded if $f(D)$ is a bounded subset of $\mathbb{R}$.*

THEOREM 40. *Let $f : D \to \mathbb{R}$ be continuous. If D is compact, then $f(D)$ is compact. (The continuous image of a compact set is compact.)*

Definition 30. *Let $f : D \to \mathbb{R}$. $f(x_0)$ is the* **minimum value of f on** *D if $f(x_0) \le f(x)$ for all $x \in D$. $f(x_1)$ is the* **maximum value of f on** *D if $f(x) \le f(x_1)$ for all $x \in D$.*

COROLLARY 1. If $f : D \to \mathbb{R}$ is continuous and D is compact, then f has a maximum value and a minimum value. That is, there exist points $x_0,\ x_1 \in D$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in D$.

COROLLARY 2. If $f : D \to \mathbb{R}$ is continuous and D is compact, then $f(D)$ is closed and bounded.

THEOREM 41. *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite sign, then there is at least one point $c \in (a,b)$ such that $f(c) = 0$.*

THEOREM 42. Intermediate Value Theorem *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Suppose that* $f(a) \neq f(b)$*. If* k *is a number between* $f(a)$ *and* $f(b)$*, then there is at least one number* $c \in (a, b)$ *such that* $f(c) = k$*.*

THEOREM 43. *If* I *is a compact interval, then* $f(I)$ *is a compact interval.*

Exercises 3.3

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.

   (a) Suppose that $f : D \to \mathbb{R}$ is continuous. then there exists a point $x_1 \in D$ such that $f(x) \leq f(x_1)$ for all $x \in D$.

   (b) If $D \subseteq \mathbb{R}$ is bounded and $f : D \to \mathbb{R}$ is continuous, then $f(D)$ is bounded.

   (c) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) \leq k \leq f(b)$. Then there exists a point $c \in [a, b]$ such that $f(c) = k$.

   (d) If $f : D \to \mathbb{R}$ is continuous and bounded on D, then f has a maximum value and a minimum value on D.

2. Let $f : D \to \mathbb{R}$ be continuous. For each of the following, prove or give a counterexample.

   (a) If D is open, then $f(D)$ is open.

   (b) If D is closed, then $f(D)$ is closed.

   (c) If D is not open, then $f(D)$ is not open.

   (d) If D is not closed, then $f(D)$ is not closed.

   (e) If D is not compact, then $f(D)$ is not compact.

   (f) If D is not bounded, then $f(D)$ is not bounded.

   (g) If D is an interval, then $f(D)$ is an interval.

   (h) If D is an interval that is not open, then $f(D)$ is an interval that is not open.

3. Prove that $\sin x + 2\cos x = x^2$ for some $x \in [0, \pi/2]$.

4. Prove that every polynomial of odd degree has at least one real root.

5. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that there is at least one point $c \in [a, b]$ such that $f(c) = c$. (Such a point is called a *fixed point of* f.)

6. Suppose that $f,\ g : [a, b] \to \mathbb{R}$ are continuous, and suppose that $f(a) \leq g(a)$, $f(b) \geq g(b)$. Prove that there is at least one point $c \in [a, b]$ such that $f(c) = g(c)$.

7. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f([a, b]) \subseteq \mathcal{Q}$ (the rational numbers). Prove that f is constant.