Chapter VI
Continuous Time Markov Chains

1. Pure Birth Processes

In this chapter we present several important examples of continuous tim discrete state, Markov processes. Specifically, we deal here with a famil of random variables $\{X(t); 0 \leq t < \infty\}$ where the possible values of $X($ are the nonnegative integers. We shall restrict attention to the case wher $\{X(t)\}$ is a Markov process with stationary transition probabilities. Thu the transition probability function for $t > 0$,

$$P_{ij}(t) = \Pr\{X(t + u) = j | X(u) = i\}, \qquad i, j = 0, 1, 2, \ldots,$$

is independent of $u \geq 0$.

It is usually more natural in investigating particular stochastic model based on physical phenomena to prescribe the so-called infinitesima probabilities relating to the process and then derive from them an explici expression for the transition probability function. For the case at hand, w will postulate the form of $P_{ij}(h)$ for h small, and, using the Markov prop erty, we will derive a system of differential equations satisfied by $P_{ij}(t)$ fo all $t > 0$. The solution of these equations under suitable boundary condi tions gives $P_{ij}(t)$.

By way of introduction to the general pure birth process, we reviev briefly the axioms characterizing the Poisson process.

1.1. Postulates for the Poisson Process

The Poisson process is the prototypical pure birth process. Let us point out the relevant properties. The Poisson process is a Markov process on the nonnegative integers for which

(i) $\Pr\{X(t+h) - X(t) = 1 | X(t) = x\} = \lambda h + o(h)$ as $h \downarrow 0$ $(x = 0, 1, 2, \ldots)$.

(ii) $\Pr\{X(t+h) - X(t) = 0 | X(t) = x\} = 1 - \lambda h + o(h)$ as $h \downarrow 0$.

(iii) $X(0) = 0$.

The precise interpretation of (i) is the relationship

$$\lim_{h \to 0+} \frac{\Pr\{X(t+h) - X(t) = 1 | X(t) = x\}}{h} = \lambda.$$

The $o(h)$ symbol represents a negligible remainder term in the sense that if we divide the term by h, then the resulting value tends to zero as h tends to zero. Notice that the right side of (i) is independent of x.

These properties are easily verified by direct computation, since the explicit formulas for all the relevant properties are available. Problem 1.13 calls for showing that these properties, in fact, define the Poisson process.

1.2. Pure Birth Process

A natural generalization of the Poisson process is to permit the chance of an event occurring at a given instant of time to depend upon the number of events that have already occurred. An example of this phenomenon is the reproduction of living organisms (and hence the name of the process), in which under certain conditions—sufficient food, no mortality, no migration, for example—the infinitesimal probability of a birth at a given instant is proportional (directly) to the population size at that time. This example is known as the *Yule process* and will be considered in detail later.

Consider a sequence of positive numbers, $\{\lambda_k\}$. We define a pure birth process as a Markov process satisfying the following postulates:

(1) $\Pr\{X(t+h) - X(t) = 1 | X(t) = k\} = \lambda_k h + o_{1,k}(h)$ $(h \to 0+)$.

(2) $\Pr\{X(t+h) - X(t) = 0 | X(t) = k\} = 1 - \lambda_k h + o_{2,k}(h)$. (1.1

(3) $\Pr\{X(t+h) - X(t) < 0 | X(t) = k\} = 0$ $(k \geq 0)$.

As a matter of convenience we often add the postulate

(4) $X(0) = 0$.

With this postulate $X(t)$ does not denote the population size but, rather, th number of births in the time interval $(0, t]$.

Note that the left sides of Postulates (1) and (2) are just $P_{k,k+1}(h)$ an $P_{k,k}(h)$, respectively (owing to stationarity), so that $o_{1,k}(h)$ and $o_{2,k}(h)$ do no depend upon t.

We define $P_n(t) = \Pr\{X(t) = n\}$, assuming $X(0) = 0$.

By analyzing the possibilities at time t just prior to time $t + h$ (h small) we will derive a system of differential equations satisfied by $P_n(t)$ fo $t \geq 0$, namely

$$\begin{aligned} P_0'(t) &= -\lambda_0 P_0(t), \\ P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \qquad \text{for } n \geq 1, \end{aligned} \tag{1.2}$$

with initial conditions

$$P_0(0) = 1, \qquad P_n(0) = 0, \qquad n > 0.$$

Indeed, if $h > 0$, $n \geq 1$, then by invoking the law of total probability, th Markov property, and Postulate (3) we obtain

$$\begin{aligned} P_n(t+h) &= \sum_{k=0}^{\infty} P_k(t) \Pr\{X(t+h) = n | X(t) = k\} \\ &= \sum_{k=0}^{\infty} P_k(t) \Pr\{X(t+h) - X(t) = n - k | X(t) = k\} \\ &= \sum_{k=0}^{n} P_k(t) \Pr\{X(t+h) - X(t) = n - k | X(t) = k\}. \end{aligned}$$

Now for $k = 0, 1, \ldots, n-2$ we have

$$\begin{aligned} &\Pr\{X(t+h) - X(t) = n - k | X(t) = k\} \\ &\quad \leq \Pr\{X(t+h) - X(t) \geq 2 | X(t) = k\} \\ &\quad = o_{1,k}(h) + o_{2,k}(h), \end{aligned}$$

or

$$\Pr\{X(t+h) - X(t) = n - k \mid X(t) = k\} = o_{3,n,k}(h), \qquad k = 0, \ldots, n-2.$$

Thus

$$P_n(t+h) = P_n(t)[1 - \lambda_n h + o_{2,n}(h)] + P_{n-1}(t)[\lambda_{n-1}h + o_{1,n-1}(h)] + \sum_{k=0}^{n-2} P_k(t) o_{3,n,k}(h)k,$$

or

$$P_n(t+h) - P_n(t) = P_n(t)[-\lambda_n h + o_{2,n}(h)] + P_{n-1}(t)[\lambda_{n-1}h + o_{1,n-1}(h)] + o_n(h), \tag{1.3}$$

where, clearly, $\lim_{h\downarrow 0} o_n(h)/h = 0$ uniformly in $t \geq 0$, since $o_n(h)$ is bounded by the finite sum $\Sigma_{k=0}^{n-2} o_{3,n,k}(h)$, which does not depend on t.

Dividing by h and passing to the limit $h \downarrow 0$, we validate the relations (1.2), where on the left side we should, to be precise, write the derivative from the right. With a little more care, however, we can derive the same relation involving the derivative from the left. In fact, from (1.3) we see at once that the $P_n(t)$ are continuous functions of t. Replacing t by $t - h$ in (1.3), dividing by h, and passing to the limit $h \downarrow 0$, we find that each $P_n(t)$ has a left derivative that also satisfies equation (1.2).

The first equation of (1.2) can be solved immediately and yields

$$P_0(t) = \exp\{-\lambda_0 t\} \qquad \text{for } t > 0. \tag{1.4}$$

Define S_k as the time between the kth and the $(k+1)$st birth, so that

$$P_n(t) = \Pr\left\{\sum_{i=0}^{n-1} S_i \leq t < \sum_{i=0}^{n} S_i\right\}.$$

The random variables S_k are called the "sojourn times" between births, and

$$W_k = \sum_{i=0}^{k-1} S_i = \text{the time at which the } k\text{th birth occurs.}$$

We have already seen that $P_0(t) = \exp\{-\lambda_0 t\}$. Therefore,

$$\Pr\{S_0 \leq t\} = 1 - \Pr\{X(t) = 0\} = 1 - \exp\{-\lambda_0 t\};$$

i.e., S_0 has an exponential distribution with parameter λ_0. It may be deduc from Postulates (1) through (4) that S_k, $k > 0$, also has an exponential di tribution with parameter λ_k and that the S_i's are mutually independent.

This description characterizes the pure birth process in terms of its s journ times, in contrast to the infinitesimal description corresponding (1.1).

To solve the differential equations of (1.2) recursively, introduce $Q_n($ $= e^{\lambda_n t} P_n(t)$ for $n = 0, 1, \ldots$. Then

$$Q_n'(t) = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t)$$
$$= e^{\lambda_n t}[\lambda_n P_n(t) + P_n'(t)]$$
$$= e^{\lambda_n t} \lambda_{n-1} P_{n-1}(t) \qquad \text{[using (1.2)].}$$

Integrating both sides of these equations and using the boundary conditi $Q_n(0) = 0$ for $n \geq 1$ gives

$$Q_n(t) = \int_0^t e^{\lambda_n x} \lambda_{n-1} P_{n-1}(x)\, dx,$$

or

$$P_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} P_{n-1}(x)\, dx, \qquad n = 1, 2, \ldots. \qquad (1.$$

It is now clear that all $P_k(t) \geq 0$, but there is still a possibility that

$$\sum_{n=0}^{\infty} P_n(t) < 1.$$

To secure the validity of the process, i.e., to assure that $\Sigma_{n=0}^{\infty} P_n(t) = 1$ f all t, we must restrict the λ_k according to the following:

$$\sum_{n=0}^{\infty} P_n(t) = 1 \quad \text{if and only if} \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty. \qquad (1.$$

The intuitive argument for this result is as follows: The time S_k betwee consecutive births is exponentially distributed with a corresponding par meter λ_k. Therefore, the quantity $\Sigma_n 1/\lambda_n$ equals the expected time befo the population becomes infinite. By comparison, $1 - \Sigma_{n=0}^{\infty} P_n(t)$ is th probability that $X(t) = \infty$.

If $\Sigma_n \lambda_n^{-1} < \infty$, then the expected time for the population to become infinite is finite. It is then plausible that for all $t > 0$ the probability that $X(t) = \infty$ is positive.

When no two of the birth parameters $\lambda_0, \lambda_1, \ldots$ are equal, the integral equation (1.5) may be solved to give the explicit formula

$$P_0(t) = e^{-\lambda_0 t},$$
$$P_1(t) = \lambda_0\left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t}\right), \tag{1.7}$$

and

$$\begin{aligned} P_n(t) &= \Pr\{X(t) = n \mid X(0) = 0\} \\ &= \lambda_0 \cdots \lambda_{n-1}[B_{0,n}e^{-\lambda_0 t} + \cdots + B_{n,n}e^{-\lambda_n t}] \qquad \text{for } n > 1, \end{aligned} \tag{1.8}$$

where

$$B_{0,n} = \frac{1}{(\lambda_1 - \lambda_0) \cdots (\lambda_n - \lambda_0)},$$
$$B_{k,n} = \frac{1}{(\lambda_0 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k)(\lambda_{k+1} - \lambda_k) \cdots (\lambda_n - \lambda_k)} \qquad \text{for } 0 < k < n, \tag{1.9}$$

and

$$B_{n,n} = \frac{1}{(\lambda_0 - \lambda_n) \cdots (\lambda_{n-1} - \lambda_n)}.$$

Because $\lambda_j \neq \lambda_k$ when $j \neq k$ by assumption, the denominator in (1.9) does not vanish, and $B_{k,n}$ is well-defined.

We will verify that $P_1(t)$, as given by (1.7), satisfies (1.5). Equation (1.4) gives $P_0(t) = e^{-\lambda_0 t}$. We next substitute this in (1.5) when $n = 1$, thereby obtaining

$$\begin{aligned} P_1(t) &= \lambda_0 e^{-\lambda_1 t}\int_0^t e^{\lambda_1 x}e^{-\lambda_0 x}\,dx \\ &= \lambda_0 e^{-\lambda_1 t}(\lambda_0 - \lambda_1)^{-1}[1 - e^{-(\lambda_0 - \lambda_1)t}] \\ &= \lambda_0\left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t}\right), \end{aligned}$$

in agreement with (1.7).

The induction proof for general n involves tedious and difficult algebr The case $n = 2$ is suggested as a problem.

1.3. The Yule Process

The Yule process arises in physics and biology and describes the grow of a population in which each member has a probability $\beta h + o(h)$ of gi ing birth to a new member during an interval of time of length h ($\beta >$ (Assuming independence and no interaction among members of the pop lation, the binomial theorem gives

$$\begin{aligned} \Pr\{X(t+h) - X(t) = 1 \mid X(t) = n\} &= \binom{n}{1}[\beta h + o(h)][1 - \beta h + o(h)]^{n-} \\ &= n\beta h + o_n(h); \end{aligned}$$

i.e., for the Yule process the infinitesimal parameters are $\lambda_n = n\beta$. words, the total population birth rate is directly proportional to the pop lation size, the proportionality constant being the individual birth rate As such, the Yule process forms a stochastic analogue of the determini tic population growth model represented by the differential equatic $dy/dt = \alpha y$. In the deterministic model, the rate dy/dt of population grow is directly proportional to population size y. In the stochastic model, t infinitesimal deterministic increase dy is replaced by the probability of unit increase during the infinitesimal time interval dt. Similar connectio between deterministic rates and birth (and death) parameters arise fr quently in stochastic modeling. Examples abound in this chapter.

The system of equations (1.2) in the case that $X(0) = 1$ becomes

$$P_n'(t) = -\beta[nP_n(t) - (n-1)P_{n-1}(t)], \qquad n = 1, 2, \ldots,$$

under the initial conditions

$$P_1(0) = 1, \qquad P_n(0) = 0, \qquad n = 2, 3, \ldots.$$

Its solution is

$$P_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \qquad n \geq 1, \tag{1.1}$$

as may be verified directly. We recognize (1.10) as the geometric distri ution I, (3.5) with $p = e^{-\beta t}$.

The general solution analogous to (1.8) but for pure birth processes starting from $X(0) = 1$ is

$$P_n(t) = \lambda_1 \cdots \lambda_{n-1}[B_{1,n}e^{-\lambda_1 t} + \cdots + B_{n,n}e^{-\lambda_n t}], \quad n > 1. \qquad (1.11)$$

When $\lambda_n = \beta n$, we will show that (1.11) reduces to the solution given in (1.10) for a Yule process with parameter β. Then

$$B_{1,n} = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)\cdots(\lambda_n - \lambda_1)} = \frac{1}{\beta^{n-1}(1)(2)\cdots(n-1)} = \frac{1}{\beta^{n-1}(n-1)!},$$

$$B_{2,n} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)\cdots(\lambda_n - \lambda_2)} = \frac{1}{\beta^{n-1}(-1)(1)(2)\cdots(n-2)} = \frac{-1}{\beta^{n-1}(n-2)!},$$

and

$$\beta_{k,n} = \frac{1}{(\lambda_1 - \lambda_k)\cdots(\lambda_{k-1} - \lambda_k)(\lambda_{k+1} - \lambda_k)\cdots(\lambda_n - \lambda_k)} = \frac{(-1)^{k-1}}{\beta^{n-1}(k-1)!(n-k)!}.$$

Thus, according to (1.11),

$$\begin{aligned} P_n(t) &= \beta^{n-1}(n-1)!(B_{1,n}e^{-\beta t} + \cdots + B_{n,n}e^{-n\beta t}) \\ &= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}(-1)^{k-1}e^{-k\beta t} \\ &= e^{-\beta t}\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!}(-e^{-\beta t})^j \\ &= e^{-\beta t}(1 - e^{-\beta t})^{n-1} \qquad \text{[see I, (6.17)]}, \end{aligned}$$

which establishes (1.10).

Exercises

1.1. A pure birth process starting from $X(0) = 0$ has birth paramet $\lambda_0 = 1, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

1.2. A pure birth process starting from $X(0) = 0$ has birth paramet $\lambda_0 = 1, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 5$. Let W_3 be the random time that it takes process to reach state 3.

(a) Write W_3 as a sum of sojourn times and thereby deduce that mean time is $E[W_3] = \frac{11}{6}$.
(b) Determine the mean of $W_1 + W_2 + W_3$.
(c) What is the variance of W_3?

1.3. A population of organisms evolves as follows. Each organism ists, independent of the other organisms, for an exponentially distribu length of time with parameter θ, and then splits into two new organisr each of which exists, independent of the other organisms, for an ex nentially distributed length of time with parameter θ, and then splits i two new organisms, and so on. Let $X(t)$ denote the number of organis existing at time t. Show that $X(t)$ is a Yule process.

1.4. Consider an experiment in which a certain event will occur w probability αh and will not occur with probability $1 - \alpha h$, where α i fixed positive parameter and h is a small ($h < 1/\alpha$) positive variable. S pose that n independent trials of the experiment are carried out, and total number of times that the event occurs is noted. Show that

(a) The probability that the event never occurs during the n trials $1 - n\alpha h + o(h)$;
(b) The probability that the event occurs exactly once is $n\alpha h + o(h$
(c) The probability that the event occurs twice or more is $o(h)$.

Hint: Use the binomial expansion

$$(1 - \alpha h)^n = 1 - n\alpha h + \frac{n(n-1)}{2}(\alpha h)^2 - \cdots.$$

1.5. Using equation (1.10), calculate the mean and variance for the Y process where $X(0) = 1$.