Name: **December 4, 2010**

FINAL

Math 5331 Linear Algebra Online
This Final is worth 200 points. You are not allowed to use any books or notes.

You have three hours to complete the test.

1. Define that **U** is a subspace of the vector space **V**. **Answer**: U is non-empty and closed under addition, that is if $u, v \in U$ then $u + v \in U$ and $\alpha.u \in U$.

2. Proof that the solution space of a linear homogeneous system $\mathbf{A}X = 0$ of m equations in n unknowns with real coefficients is a subspace of R^n **Answer**: We always have $AO = 0$, that is the solution space is never empty. And $AX = 0$ and $AY = 0$, then $A(X+Y) = AX + AY = 0 + 0 = 0, A\alpha X = \alpha AX = \alpha 0 = 0$

3. **a.** Define that the vectors $v_1, v_2, \ldots, v_k$ are linearly independent. Answer: $c_1 v_1 + \ldots c_k v_k = 0$ only if $c_1 = \ldots = c_k = 0$

b. Define the span of vectors $v_1, v_2, \ldots, v_k$. Answer: $span\{v_1, \ldots v_k\} = \{c_1 v_1 + \ldots + c_k v_k | c_1 \in \mathbb{R} .. c_k \in \mathbb{R}\}$ The span is the set of all linear combinations.

4. Define that the vectors $v_1, v_2, \ldots, v_k$ are a basis of the vector space **V**. Answer: The vectors have to be linearly independents and generating.

5. Find a basis of all solutions for the linear homogeneous system

$$\begin{pmatrix} 1 & -1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0$$

Answer: The matrix is already in row echelon form
$x_1 = x_2 - 5x_4, x_3 = -2x_4 - 2x_6, x_5 = -2x_6$. Free variables are x_2, x_4, x_6. This gives us a three dimensional solution space:
$X_2 = (1, 1, , 0, 0, 0), X_4 = (-5, 0, -2, 1, 0, 0), X_6 = (0, 0, -2, 0, -2, 1)$ are a basis. SNB check:

$$\begin{pmatrix} 1 & -1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \text{null-space basis: } \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right]$$

6. If $\mathbf{A}$ is an $m \times n$ matrix of rank r, what is then the dimension of the solution space of the linear homogeneous system $\mathbf{A}X = 0$? Answer: $n - r$

7. **a**. Define that $T : \mathbf{U} \to \mathbf{V}$ is a linear map from the vector space $\mathbf{U}$ to the vector space $\mathbf{V}$. Answer: $T(\alpha + \beta) = T(\alpha) + T(\beta), T(c\alpha) = cT(\alpha)$

b. How are null-spaces and range of a linear map defined?Answer: $N(T) = \{\alpha | T(\alpha) = 0\}, R(T) = \{\beta | \beta = T(\alpha) \text{ for some } \alpha\}$

c. Let T be a linear map from R^n to R^m. How is the matrix $\mathbf{A}$ for T with respect to the unit vectors defined? Answer: The matrix A for T is an $m \times n$ – matrix where you have in the i^{th} – column the $m - tuple\ T(e_i^n) \in \mathbb{R}^m, i = 1, \ldots n$.

d. Express null space and range of T in terms of the matrix $\mathbf{A}$ for T. In particular relate column and row rank of $\mathbf{A}$ to the dimensions of null space and range of T. Answer: The null-space of T is the solution space for $AX = 0$. The range of T is the space generated by the columns of A. We have *column rank=dimension of the range, dimension of null space=n – row rank*.

8. Let $T : \mathbf{U} \to \mathbf{V}$ be a linear map from the vector space $\mathbf{U}$ to the vector space $\mathbf{V}$. Prove that if the vectors $T(v_1), T(v_2), \ldots, T(v_k)$ are linearly independent, then $v_1, v_2, \ldots, v_k$ are linearly independent. Answer: We can prove this in contraposition. Assume that $v_1, v_2, \ldots, v_k$ are linearly dependent. Then without loss of generality one has that $c_1 = c_2v_2 + \ldots + c_kv_k$. But then $T(c_1) = c_2T(v_2)\ldots + c_kT(v_k)$ that is $T(c_1), \ldots, T(c_k)$ are linearly dependent.

9. Assume that the linear map T on R^3 has matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 & 7 \\ 1 & 3 & 5 \end{pmatrix}$$

a. Find a basis for the null space of T. Answer: null-space basis $(-2, -1, 1)$ Trivial calculation

b. Find a basis for the range of T. Answer: any two columns are are basis.

10. Let T be a linear map from $\mathsf{R}^n \to \mathsf{R}^n$ and let $\lambda \in \mathsf{R}$.

a. Define the characteristic polynomial $c_T(x)$ of T. What is the characteristic polynomial of the identity map on R^n and what is the characteristic polynomial of the zero map on R^n? Answer: Let A be the matrix of $A. c_T(x) = \det(A - xI_n); c_{I_n}(x) = \det(I_n - xI_n) = (1 - x)^n; \det(0_{n\times n} - xI_n) = (-1)^n x^n$

b. Define that λ is an eigenvalue of T. Answer: $c_T(\lambda) = 0$;

c. Define that $\mathbf{E}_\lambda$ is the eigenspace for λ. Answer: $E_\lambda = \{v | T(v) = \lambda v\}$