Homework 2 Logic with Applications

1. A partially ordered set \((L, \leq)\) is a complete lattice if every subset has an infimum, i.e., a largest lower bound. Show that in a complete lattice every subset has a supremum.

2. Let \(\mathcal{S}\) be a system of subsets of a set \(A\). Assume that \(\mathcal{S}\) is closed under intersections. Show that \((\mathcal{S}, \subseteq)\) is a complete lattice. Examples?

3. Let \(A\) be any set. A system \(\mathcal{C}\) of subsets of \(A\) is called a closure system if \(\mathcal{C}\) is closed under arbitrary intersections. That is, if \(D \subseteq \mathcal{C}\) then \(\bigcap D \in \mathcal{C}\). In particular, taking \(D = \emptyset\), yields \(A \in \mathcal{C}\). According to the previous problems, we know that \((\mathcal{C}, \subseteq)\) is a complete lattice with \(\bigcap \mathcal{C}\) as the smallest element and \(A\) as the largest element. Related to closure systems are the closure operators. These are the maps \(C : \mathcal{P}(A) \to \mathcal{P}(A)\) subject to the following conditions:

   (C1) \(C(X) \subseteq C(Y)\) in case that \(X \subseteq Y\). That is, \(C\) is monotone.

   (C2) \(X \subseteq C(X)\). That is, \(C\) is extensive.

   (C3) \(C(C(X)) = C(X)\). That is, \(C\) is idempotent.

   Show: If \(C\) is a closure system then
   \[
   C_C(X) = \bigcap \{Y | Y \in \mathcal{C}, X \subseteq Y\}, X \in \mathcal{P}(A)
   \]
   defines a closure operator. Conversely, if \(C\) is a closure operator then
   \[
   C_C = \{X | X \subseteq A, C(X) = X\}
   \]
   is a closure system. Prove: \(C \mapsto C_C, C \mapsto C_C\) establishes a bijective correspondence between the closure systems and the closure operators of the set \(A\). For a given closure operator \(C\), the elements of \(C_C\) are called the closed sets of \((A, C)\). For a given closure system \(C\), the set \(C_C(X)\) is called the \(C\)-closure of \(X\).

4. Let \(A\) and \(B\) be any sets and let \(R\) be a binary relation between the elements of \(A\) and the elements of \(B\), i.e., \(R \subseteq A \times B\). The polarities of the relation \(R\) are the mappings between the power sets \(\mathcal{P}(A)\) and \(\mathcal{P}(B)\) that are defined by
   \[
   U \mapsto U^* = \{b | (u, b) \in R \text{ for all } u \in U\}
   \]
   \[
   V \mapsto V^* = \{a | (a, v) \in R \text{ for all } v \in V\}
   \]
   The polarities satisfy:

   (G1) \(U_1^* \subseteq U_2^*\) in case that \(U_1 \supseteq U_2\); \(V_1^* \subseteq V_2^*\) in case that \(V_1 \supseteq V_2\).
(G2) $U \subseteq U^{**}, V \subseteq V^{**}$.

A pair of mappings $\_^* : \mathcal{P}(A) \to \mathcal{P}(B), \_^* : \mathcal{P}(B) \to \mathcal{P}(A)$ that satisfies (G1) and (G2) is called a Galois connection. Prove that you have for any Galois connection also:

(G3) $U^{***} = U^*$ and $V^{***} = V^*$.

Deduce that for any Galois connection one has that $U \mapsto U^{**}$ and $V \mapsto V^{**}$ are closure operators and that a subset of $A$ is closed iff it is of the form $V^*$ for some subset $V$ of $B$ and that the lattice of closed subsets of $A$ is order isomorphic to the dual of the lattice of closed subsets of $B$.

Relate syntax and semantic in logic to polarities and Galois connections.