Hall's Theorem for Propositional Formulas

The set of propositional connectives is $\mathcal{C} = \{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$. The *arities* of these connectives are 1 for negation $\neg$, and 2 for all the others. The set of propositional variables is $\mathcal{P} = \{p, q, r, \ldots\}$. A *word* is a non-empty string of symbols taken from $\mathcal{C} \cup \mathcal{P}$. Certain words are formulas, $\vee pq$, others are not, e.g., $p \vee q$. Note, that we are going to use Polish notation which is due to J. Łukasiewicz.

The following is an algorithm due to P. Hall. First, we define the *length* $l(w)$ of a word w as the number of characters in it. E.g., if $w = x_1 x_2 \ldots x_n$ then $l(w) = n$. Thus, $l(ww') = l(w) + l(w')$.

Next, we define the *valency* $v(w)$ of a word w. There are two types of words of length one, connectives and variables. If c is a connective then we define $v(c) =$ (-n+1) where n is the arity of c. Thus, $v(\neg) = 0$ and $v(c) = -1$ for all the other connectives. If p is a variable then we define $v(p) = 1$. Then, if $w = x_1 \ldots x_n$ we define $v(w) = v(x_1) + \ldots + v(x_n)$. Again, $v(ww') = v(w) + v(w')$.

For example, $v(\vee pq) = v(\vee) + v(p) + v(q) = (-2 + 1) + 1 + 1 = 1$.

We may think that we input from right to left the variables q and p which yields valency 2 and then the $\vee$ absorbs the two variables yielding a proper formula of valency 1.

Now prove the following theorem by induction on $l(w)$:

Theorem. *The word w is a product of k formulas, i.e.,*

$$w = w_1 \ldots w_k$$

if and only if

$$v(w) = k \text{ and } v(u) > 0 \text{ for every right segment } u \text{ of } w.$$

Corollary. *A word w is a formula if and only if $v(w) = 1$ and $v(u) > 0$ for every right segment u of w.*

We also have the

Formation Lemma. *Assume that w is a product of formulas w_i, w'_j:*

$$w = w_1 \ldots w_n = w'_1 \ldots w'_m$$

then $m = n$ and $w_i = {w'}_i$.