Hall’s Theorem for Propositional Formulas

The set of propositional connectives is $C = \{\neg, \lor, \land, \rightarrow, \leftrightarrow\}$. The arities of these connectives are 1 for negation $\neg$, and 2 for all the others. The set of propositional variables is $P = \{p, q, r, \ldots\}$. A word is a non-empty string of symbols taken from $C \cup P$. Certain words are formulas, e.g., $p \lor q$, others are not, e.g., $p \lor q$.

Note, that we are going to use Polish notation which is due to J. Lukasiewicz.

The following is an algorithm due to P. Hall. First, we define the length $l(w)$ of a word $w$ as the number of characters in it. E.g., if $w = x_1x_2 \ldots x_n$ then $l(w) = n$. Thus, $l(ww') = l(w) + l(w')$.

Next, we define the valency $v(w)$ of a word $w$. There are two types of words of length one, connectives and variables. If $c$ is a connective then we define $v(c) = (-n+1)$ where $n$ is the arity of $c$. Thus, $v(\neg) = 0$ and $v(\lor) = -1$ for all the other connectives. If $p$ is a variable then we define $v(p) = 1$. Then, if $w = x_1 \ldots x_n$ we define $v(w) = v(x_1) + \ldots + v(x_n)$. Again, $v(ww') = v(w) + v(w')$.

For example, $v(\lor pq) = v(\lor) + v(p) + v(q) = (-2 + 1) + 1 + 1 = 1$.

We may think that we input from right to left the variables $q$ and $p$ which yields valency 2 and then the $\lor$ absorbs the two variables yielding a proper formula of valency 1.

Now prove the following theorem by induction on $l(w)$:

**Theorem.** The word $w$ is a product of $k$ formulas, i.e.,

$$w = w_1 \ldots w_k$$

if and only if

$$v(w) = k \text{ and } v(u) > 0 \text{ for every right segment } u \text{ of } w.$$  

**Corollary.** A word $w$ is a formula if and only if $v(w) = 1$ and $v(u) > 0$ for every right segment $u$ of $w$.

We also have the

**Formation Lemma.** Assume that $w$ is a product of formulas $w_i, w'_j$:

$$w = w_1 \ldots w_n = w'_1 \ldots w'_m$$

then $m = n$ and $w_i = w'_i$. 

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