Closure Operators and Galois Connections

1. A collection $\mathcal{C}$ of subsets of a set A is a *closure system* if it is closed under arbitrary intersections: If $\mathcal{S} \subseteq \mathcal{C}$ then $\bigcap \mathcal{S} \in \mathcal{C}$. In particular, one has that $A \in \mathcal{C}$, as the empty intersection.
   $C : \mathcal{P} \to \mathcal{P}$ is called a *closure operator* on the set A if it is *extensive*, *monotone* and *idempotent*, i.e., one has for subsets B of A:

   (a) $B \subseteq C(B)$;

   (b) if $B_1 \subseteq B_2$ then $C(B_1) \subseteq (C(B_2)$;

   (c) $C(B) = C(C(B))$.

   Show : For every closure system $\mathcal{C}$, $C(B) = \bigcap\{M | M \supseteq B, M \in \mathcal{C}\}$ is a closure operator, and for every closure operator C, the system $\mathcal{C} = \{M | M = C(M)\}$ is a closure system and this defines a bijection between closure systems and closure operators.

2. Let $(S, \leq)$ and $(T, \leq)$ be posets. A pair (g, d) of functions $g : S \to T, d : T \to S$ is called a *Galois connection* if (i) g and d are monotone; (ii) $g(s) \geq t$ iff $s \geq d(t)$ holds for all $s \in S$ and $t \in T$. Prove:

   (a) The *upper adjoint* g preserves all infima in S and the *lower adjoint* d all suprema in T.

   (b) g is surjective iff $g \circ d = \mathrm{id}_T$ iff d is injective.

   (c) g is injective iff $d \circ g = \mathrm{id}_S$ iff d is surjective.

   (d) $c = g \circ d : T \to T$ is a closure operator on $(T, \leq)$, while $k = d \circ g : S \to S$ is a *kernel operator* on $(S, \leq)$, that is, k is *intensive* ($s \geq k(s)$), monotone and idempotent.
   Note: $g \circ d \circ g = g, d \circ g \circ d = d$.

   (e) Let $R \subseteq A \times B$ be a binary relation. Then:

      i. $g : (\mathcal{P}(A), \subseteq^*) \to (\mathcal{P}(A), \subseteq), X \mapsto \{y | (x, y) \in R$ for all $x \in X\} = X^*$,

      ii. $d : (\mathcal{P}(B), \subseteq) \to (\mathcal{P}(A), \subseteq^*), Y \mapsto \{x | (x, y) \in R$ for all $y \in Y\} = Y^*$.

      defines a Galois connection.

   Examples?