June 21, 2008

Name:

1. Express in the form $P \to Q$ or $Q \to P$ the following statements:

a) P in case that Q. $Q \to P$

b) P only if Q $P \to Q$

c) P if Q. $Q \to P$

d) P is necessary for Q. $Q \to P$

e) P is sufficient for Q. $P \to Q$

f) P whenever Q $Q \to P$

2. State whether the formula is a tautology or not.

a. $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$ **tautology**:
$(p \to (q \to r)) \leftrightarrow \neg p \vee (\neg q \vee r) \leftrightarrow \neg p \vee \neg q \vee r \leftrightarrow \neg(p \wedge q) \vee r \leftrightarrow p \wedge q \to r$

b. $((p \wedge \neg p) \to q)$ **tautology**, $((p \wedge \neg p) \to q) \leftrightarrow F \to q$ is a tautology

c. $(p \to (q \to r)) \to ((p \to q) \to r))$ **not a tautology**: $p \equiv F, r \equiv F$ makes the right hand side false, but the left-hand side true.

d. $(((p \vee q) \wedge \neg p) \to q$ **tautology**, cannot be made false: in order to be false we need $q \equiv F$ and $(p \vee q) \wedge \neg p$ be true. Thus $q \equiv F$ but then $p \equiv T$ and then $(p \vee q) \wedge \neg p$ is false.

2. Determine whether the following arguments are valid or invalid. $r \equiv$ raining, $h \equiv$ stay home; $c(x) \equiv x$ is citizen; $v(x) \equiv x$ can vote

a. If it is raining, then I will stay home. It is not raining. Thus I will not stay home.
$r \to h, \neg r;$ thus $\neg h$ **false**

b. If it is raining, then I will stay home. I am not staying home. Thus it is not raining
$r \to h, \neg h;$ thus $\neg r$ **correct**

c. I will stay home only if it is raining. It is not raining. Thus I will not stay home.
$h \to r, \neg r;$ thus $\neg h$ **correct**.

d. Only citizens can vote. Paul cannot vote. So he is not a citizen.
$v(x) \to c(x), \neg v(paul);$ thus $\neg c(paul)$ **false (paul might be too young to vote)**

3. True or false:

a. A function f from the set A to the set B is injective (or one-tone) if $f(x) \neq f(y)$ implies $x \neq y$. **incorrect**

b. A function f from the set A to the set B is injective (or one-tone) if $f(x) \neq f(y)$ in case that $x \neq y$. **correct**

4. Find a function $f : \mathrm{N} \to \mathrm{N}$ which is injective but not surjective and a function $g : \mathrm{N} \to \mathrm{N}$ which is surjective but not injective. Solution: $f(n) = 2n$ is an injective function but not surjective. $g(n) =$ number of primes that divide n is surjective but not injective

5. Let A be a **finite** set. Say A has five elements.

a. Can you find a function $g : A \to A$ which is injective but not surjective? Explain your answer.

b. Can you find a function $f : A \to A$ which is surjective but not injective? Explain your answer. **There is no such function. A functions on a finite set of n –many elements is injective if different elements have different images. thus there must be n –many images. thus f is surjective . And if f is surjective then there are n –many**

images thus no two elements can have the same image. thus *f* is surjective.

6. Are there any subsets of the empty set $\emptyset$? **yes, the empty set is a subset of the empty set.**
7. Let B be a set such that for every set A you have that $A \cup B = A$. What can you say about B? $B = \emptyset$
8. What is the number of elements of the set $\{\emptyset, \{\{\emptyset\}\}\}$?4
9. State the Method of Mathematical Induction and prove by induction that for $n \geq 1$ one has that $n < 2^n$. For $n = 0$ we have $0 < 2^0 = 1$; Assume that $n < 2^n$. Then $2n < 2 \cdot 2^n = 2^{n+1}$. But then $n < 2n < 2^{n+1}$