Linear Systems and the Span of Vectors

Given a set S of vectors, the problem of finding a basis for the span is an important problem in Linear Algebra. It has a straight-forward solution. If $S = \{v_1, v_2, \ldots, v_k\}$ then let A the matrix which has the v_i as its rows. It is easy to see that the elementary row operations on A do not change the $span(S)$. That is, the non-zero rows of $REF(A)$ form a basis and its number gives us also $\dim(span(S)$. The following example should look familiar:

$S = \{v_1 = (1,0,-1,0),\ v_2 = (0,1,1,1), v_3 = (5,4,-1,4)\}$

Then $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & -1 & 4 \end{pmatrix}$ has the row echelon form: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus, $u_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix}$ form a basis and $\dim(span(S) = 2$.

Because v_1 and v_2 are linearly independent, they too form a basis. Or any two of the three $v_i's$.

One could also start with the observation that the first two vectors are linearly independent and check for $v_3 \in span\{v_1, v_2\}$. That amounts to solving

$x_1 v_1 + x_2 v_2 = v_3$. This is a linear system :$x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \\ 4 \end{pmatrix}$. It can be solved now by computing the row echelon form of $B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix}$ This is a 4×2 – matrix with an augmented right column.

Again, an easy calculation yields:

$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix}$ has row echelon form: $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and this means, $x_1 = 5$ and

$x_2 = 4$. Indeed, $5\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \\ 4 \end{pmatrix}$

Coming back to A, what is the meaning of $AX = 0$? It stands for the homogeneous system:
$x_1 - x_3 = 0, x_2 + x_3 + x_4 = 0, 5x_1 + 4x_2 - x_3 + 4x_4 = 0.$ From its row echelon form we see that $x_1 = x_3, x_2 = -x_3 - x_4$ and we get the general solution X as linear combinations of two basis solutions:

$$X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, X = x_3\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Now comes the question: What have X_3 and X_4 to do with $span(S)$?? These two vectors span a two-dimensional subspace of $\mathbb{R}^4$, but certainly not $span(S)$.
What we have is this: $v_1X_3 = 0, v_1X_4 = 0; v_2X_1 = 0, v_2X_2 = 0$ (we can omit similar equations with v_3) The products are matrix products of rows $v_i's$ and columns X_j. Of course, $v_iX_j = 0$ is the same as $X_j^t v_i^t = 0$ This says that the system $CX = 0$ where the rows are X_3^t, X_4^t has as $span(S)$ as its solution space:
$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} X = 0$ has solution space $span(S)$. Check:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This example can be generalized to a theorem.

Theorem Any subspace U of $\mathbb{R}^n$ is the solution set of a linear system $CX = 0$.

Proof. Let $S = \{\alpha_1, \ldots, \alpha_k\}$ be a basis of U. Then solve the linear system $AX = 0$ which has the k –vectors $\alpha_i \in \mathbb{R}^n$ as rows. A is an $k \times n$ –matrix. The rank of B is k and therefore the null-space is of dimension $n - k$. We get $(n - k)$ –many linearly independent solution vectors $X_1, \ldots, X_{n-k}$. These are column vectors with n –many components. We form the matrix C which has the X_i as rows. Then C has rank $n - k$ and therefore the null space has dimension k. This null space contains S and therefore is equal to U.

Example.
Let $S = \{u = (1,0,1,2,3),(2,1,3,4,-1)\}$ be two vectors in $\mathbb{R}^5$. They are obviously independent and the span is 2 –dimensional.
$U = \{su + tv \mid s,t \in \mathbb{R}\}$ is a plane in $\mathbb{R}^5$. Then let

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 4 & -1 \end{pmatrix}$$, A has nullspace basis: $$\left[\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

Then let C be the matrix which has these 3 –columns as rows:

$$C = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ -3 & 7 & 0 & 0 & 1 \end{pmatrix}$$, has nullspace basis: $$\begin{pmatrix} \frac{1}{2} \\ \frac{3}{14} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{7} \\ -\frac{1}{7} \\ 0 \\ 1 \end{pmatrix}$$

It is not immediate that U has these two vectors as basis. We want to confirm that there are s,t such that

$$s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{14} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}$$ The augmented matrix of this linear system is

$$\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{3}{14} \\ 1 & 3 & \frac{5}{7} \\ 2 & 4 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$ and this system is solved via the row echelon form:

$$\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{3}{14} \\ 1 & 3 & \frac{5}{7} \\ 2 & 4 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$, row echelon form: $$\begin{pmatrix} 1 & 0 & \frac{1}{14} \\ 0 & 1 & \frac{3}{14} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ That is

$$\frac{1}{14}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} + \frac{3}{14}\begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{14} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}$$ which is correct.

And similarly, $$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{7} \\ 1 & 3 & -\frac{1}{7} \\ 2 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}$$ has row echelon form: $$\begin{pmatrix} 1 & 0 & \frac{2}{7} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ which should

give $$\frac{2}{7}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{7} \\ -\frac{1}{7} \\ 0 \\ 1 \end{pmatrix}$$

So, indeed, C is a matrix for U in the sense that U is the solution space for $CX = 0$

We also can interpret $CX = 0$ as finding a base for the kernel of the linear map T : $X \mapsto CX$ and we get the

Theorem Any subspace U is the kernel of some linear map T.