Linear Systems and the Span of Vectors

Given a set $S$ of vectors, the problem of finding a basis for the span is an important problem in Linear Algebra. It has a straight-forward solution. If $S = \{v_1, v_2, \ldots, v_k\}$ then let $A$ the matrix which has the $v_i$ as its rows. It is easy to see that the elementary row operations on $A$ do not change the span($S$). That is, the non-zero rows of $REF(A)$ form a basis and its number gives us also $\dim$span($S$). The following example should look familiar:

$S = \{v_1 = (1,0,-1,0), v_2 = (0,1,1,1), v_3 = (5,4,-1,4)\}$

Then $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & -1 & 4 \end{pmatrix}$ has the row echelon form:

$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus, $u_1 = (1 0 -1 0)$ and $u_2 = (0 1 1 1)$ form a basis and $\dim$span($S$) = 2.

Because $v_1$ and $v_2$ are linearly independent, they too form a basis. Or any two of the three $v_i$'s.

One could also start with the observation that the first two vectors are linearly independent and check for $v_3 \in$ span$\{v_1, v_2\}$. That amounts to solving

$x_1v_1 + x_2v_2 = v_3. This is a linear system \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \\ 4 \end{pmatrix}. It can be solved now by computing the row echelon form of $B = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix}$ This is a $4 \times 2$ - matrix with an augmented right column.

Again, an easy calculation yields:

$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix}$ has row echelon form: $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and this means, $x_1 = 5$ and
$x_2 = 4$. Indeed, $5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \\ 4 \end{pmatrix}$

Coming back to $A$, what is the meaning of $AX = 0$? It stands for the homogeneous system:

$x_1 - x_3 = 0, x_2 + x_3 + x_4 = 0, 5x_1 + 4x_2 - x_3 + 4x_4 = 0$. From its row echelon form we see that $x_1 = x_3, x_2 = -x_3 - x_4$ and we get the general solution $X$ as linear combinations of two basis solutions:

$X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, X = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix}$

Now comes the question: What have $X_3$ and $X_4$ to do with $\text{span}(S)$? These two vectors span a two-dimensional subspace of $\mathbb{R}^4$, but certainly not $\text{span}(S)$. What we have is this: $v_1X_3 = 0, v_1X_4 = 0; v_2X_1 = 0, v_2X_2 = 0$ (we can omit similar equations with $v_3$). The products are matrix products of rows $v_i$'s and columns $X_j$. Of course, $v_iX_j = 0$ is the same as $X_jv_i = 0$. This says that the system $CX = 0$ where the rows are $X_3, X_4$ has as $\text{span}(S)$ as its solution space:

$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}X = 0$ has solution space $\text{span}(S)$. Check:

$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ -1 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This example can be generalized to a theorem.

**Theorem** Any subspace $U$ of $\mathbb{R}^n$ is the solution set of a linear system $CX = 0$.

Proof. Let $S = \{\alpha_1, \ldots, \alpha_k\}$ be a basis of $U$. Then solve the linear system $AX = 0$ which has the $k$-vectors $\alpha_j \in \mathbb{R}^n$ as rows. $A$ is an $k \times n$-matrix. The rank of $B$ is $k$ and therefore the null-space is of dimension $n - k$. We get $(n - k)$-many linearly independent solution vectors $X_1, \ldots, X_{n-k}$. These are column vectors with $n$-many components. We form the matrix $C$ which has the $X_j$ as rows. Then $C$ has rank $n - k$ and therefore the null space has dimension $k$. This null space contains $S$ and therefore is equal to $U$. 


Example.
Let $S = \{ u = (1,0,1,2,3), (2,1,3,4,-1) \}$ be two vectors in $\mathbb{R}^5$. They are obviously independent and the span is 2-dimensional.

$U = \{ su + tv \mid s,t \in \mathbb{R} \}$ is a plane in $\mathbb{R}^5$. Then let

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 4 & -1 \end{pmatrix}, A \text{ has nullspace basis:} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then let $C$ be the matrix which has these 3 columns as rows:

$$C = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ -3 & 7 & 0 & 0 & 1 \end{pmatrix}, \text{ has nullspace basis:} \begin{pmatrix} \frac{1}{2} \\ \frac{3}{14} \\ \frac{5}{7} \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{7} \\ -\frac{1}{7} \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

It is not immediate that $U$ has these two vectors as basis. We want to confirm that there are $s,t$ such that

$$s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{14} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}$$

The augmented matrix of this linear system is

$$\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{3}{14} \\ 1 & 3 & \frac{5}{7} \\ 2 & 4 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

and this system is solved via the row echelon form:
\[
\begin{pmatrix}
1 & 2 & \frac{1}{2} \\
0 & 1 & \frac{3}{14} \\
1 & 3 & \frac{5}{7} \\
2 & 4 & 1 \\
3 & -1 & 0
\end{pmatrix}, \text{ row echelon form: } \\
\begin{pmatrix}
1 & 0 & \frac{1}{14} \\
0 & 1 & \frac{3}{14} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{That is}
\]
\[
\begin{pmatrix}
\frac{1}{14} \\
0 \\
1 \\
2 \\
3
\end{pmatrix} + \frac{3}{14}
\begin{pmatrix}
2 \\
1 \\
3 \\
4 \\
-1
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{2} \\
\frac{3}{14} \\
\frac{4}{7} \\
1 \\
0
\end{pmatrix}
\]
which is correct.

And similarly,
\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & -\frac{1}{7} \\
1 & 3 & -\frac{1}{7} \\
2 & 4 & 0 \\
3 & -1 & 1
\end{pmatrix}
\]
has row echelon form:
\[
\begin{pmatrix}
1 & 0 & \frac{2}{7} \\
0 & 1 & -\frac{1}{7} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
which should
\[
\begin{pmatrix}
1 \\
0 \\
1 \\
2 \\
3
\end{pmatrix} - \frac{1}{7}
\begin{pmatrix}
2 \\
1 \\
3 \\
4 \\
-1
\end{pmatrix} = 
\begin{pmatrix}
0 \\
-\frac{1}{7} \\
-\frac{1}{7} \\
0 \\
1
\end{pmatrix}
\]
So, indeed, \( C \) is a matrix for \( U \) in the sense that \( U \) is the solution space for \( CX = 0 \).

We also can interpret \( CX = 0 \) as finding a base for the kernel of the linear map \( T : X \mapsto CX \) and we get the

**Theorem** Any subspace \( U \) is the kernel of some linear map \( T \).